Separation of Variables Integration

1) $\frac{dy}{dx} = \frac{y-1}{x^2}$ $f(2) = 0, find\ y.$

$$\frac{dy}{y-1} = \frac{dx}{x^2} \quad \int \frac{dy}{y-1} = \int \frac{dx}{x^2}$$

$$ln|y-1| = -\frac{1}{x} + C \quad f(2) = 0$$

$$ln|0-1| = -\frac{1}{2} + C \quad 0 = -\frac{1}{2} + C \quad C = \frac{1}{2}$$

$$|y-1| = e^{-\frac{1}{x}+\frac{1}{2}}$$

Either $y - 1 = e^{-\frac{1}{x}+\frac{1}{2}}$ or $y - 1 = -e^{-\frac{1}{x}+\frac{1}{2}}$

Checking

$-1 = e^{-\frac{1}{2}+\frac{1}{2}}$	$-1 = -e^{-\frac{1}{2}+\frac{1}{2}}$
$-1 = e^0 = 1$	$-1 = -e^0 = -1$
Wrong	Yes

$$y - 1 = -e^{-\frac{1}{x}+\frac{1}{2}} \rightarrow y = 1 - e^{-\frac{1}{x}+\frac{1}{2}}$$

2) $\frac{dy}{dx} = \frac{x+1}{y}, f(0) = -2, find\ y.$

$ydy = (x+1)dx$ $\int ydy = \int (x+1)dx$ $\frac{y^2}{2} = \frac{x^2}{2} + x + C$

$y^2 = x^2 + 2x + C$ $f(0) = -2$ $4 = C$ $y^2 = x^2 + 2x + 4$

$y = -\sqrt{x^2 + 2x + 4}$

3) $f'(x) = (4 - x)x^{-3}, f(1) = 2, find\ f.$

$$\frac{dy}{dx} = (4 - x)x^{-3} \quad dy = (4x^{-3} + x^{-2})dx$$

$$\int dy = \int (4x^{-3} + x^{-2})d\,x$$

$$y = -\frac{4x^{-2}}{-2} - x^{-1} + C \to y = \frac{2}{x^2} - \frac{1}{x} + C \quad f(1) = 2$$

$$2 = \frac{2}{1} - \frac{1}{1} + C \;\; 2 = 2 - 1 + C \;\; 1 = C$$

$$y = \frac{2}{x^2} - \frac{1}{x} + 2$$

4) $\frac{dW}{dt} = \frac{1}{25}(W - 300), W(0) = 1400, find\ W.$

$\frac{dW}{w-300} = \frac{1}{25}dt \qquad \int \frac{dW}{w-300} = \int \frac{dt}{25}$

$ln|W-300| = \frac{1}{25}t + C$

$ln1100 = C$

$|W-300| = e^{\frac{t}{25}+ln1100}$

$W-300 = e^{\frac{t}{25}+ln1100}$ OR $W-300 = -e^{\frac{t}{25}+ln1100}$

$W = 300 + e^{\frac{t}{25}+ln1100}$ OR $W = 300 - e^{\frac{t}{25}+ln1100}$

Check

$1400 = 300 + 1100e^0 \qquad 1400 = 300 - 1100e^0$

Yes. No.

$W = 300 + 1100e^{\frac{t}{25}}$

5) $\frac{dy}{dx} = xy^3, f(1) = 2$, find y.

$$\frac{dy}{y^3} = xdx \quad \int \frac{dy}{y^3} = \int xdx \quad \frac{y^{-2}}{-2} = \frac{x^2}{2} + C$$

$$y^{-2} = -x^2 + C \quad \frac{1}{y^2} = -x^2 + C \quad f(1) = 2 \quad \frac{1}{4} = -1 + C \quad C = \frac{5}{4}$$

$$\frac{1}{y^2} = -x^2 + \frac{5}{4} \quad \frac{1}{y^2} = \frac{-4x^2+5}{4} \quad y^2 = \frac{4}{-4x^2+5}$$

$$y = \sqrt{\frac{4}{-4x^2+5}}$$

6) $\frac{dy}{dx} = (y-1)^2 \cos(\pi x), f(1) = 0, find\ y.$

$$\frac{dy}{(y-1)^2} = \cos(\pi x)\,dx \qquad \int \frac{dy}{(y-1)^2} = \int \cos(\pi x)\,dx$$

$$-\frac{1}{y-1} = \frac{\sin\pi x}{\pi} + C \quad -\frac{1}{-1} = 0 + C \quad 1 = C$$

$$-\frac{1}{y-1} = \frac{\sin\pi x}{\pi} + 1 \qquad -\frac{1}{y-1} = \frac{\sin\pi x + \pi}{\pi}$$

$$\frac{1}{y-1} = -\frac{\sin\pi x + \pi}{\pi} \qquad y - 1 = -\frac{\pi}{\sin\pi x + \pi}$$

$$y = 1 - \frac{\pi}{\sin\pi x + \pi}$$

7) $\frac{dy}{dx} = -\frac{xy^2}{2}, f(-1) = 2$, find y.

$$\frac{dy}{y^2} = -\frac{xdx}{2} \qquad \int \frac{dy}{y^2} = \int -\frac{xdx}{2} \qquad -\frac{1}{y} = -\frac{x^2}{4} + C$$

$$-\frac{1}{2} = -\frac{1}{4} + C \qquad C = -\frac{1}{4} \qquad -\frac{1}{y} = -\frac{x^2}{4} - \frac{1}{4}$$

$$-\frac{1}{y} = \frac{-(x^2+1)}{4} \qquad y = \frac{4}{x^2+1}$$

8) $\frac{dy}{dx} = -\frac{2x}{y}, f(1) = -1$, find y.

$$y\,dy = -2x\,dx \qquad \int y\,dy = -\int 2x\,dx \qquad \frac{y^2}{2} = -x^2 + C$$

$$y^2 = -2x^2 + C \qquad f(1) = -1 \qquad (-1)^2 = -2(1)^2 + C$$

$$1 = -2 + C \qquad 3 = C \qquad y^2 = -2x^2 + 3 \qquad y = -\sqrt{-2x^2 + 3}$$

9) $\frac{dy}{dx} = x^4(y-2), f(0) = 0$, find y.

$$\frac{dy}{y-2} = x^4 dx \qquad \int \frac{dy}{y-2} = \int x^4 dx \qquad ln|y-2| = \frac{x^5}{5} + C$$

$$ln2 = 0 + C \quad ln2 = C$$

$$ln|y-2| = \frac{x^5}{5} + ln2 \qquad |y-2| = e^{\frac{x^5}{5}+ln2}$$

$$|y-2| = 2e^{\frac{x^5}{5}}$$

$$y-2 = 2e^{\frac{x^5}{5}} \qquad OR \qquad y-2 = -2e^{\frac{x^5}{5}}$$

$$y = 2 + 2e^{\frac{x^5}{5}} \qquad\qquad y = 2 - 2e^{\frac{x^5}{5}}$$

$$0 = 2 + 2e^0 = 4 \qquad\qquad 0 = 2 - 2e^0 = 0$$

NO Yes

$$y = 2 - 2e^{\frac{x^5}{5}}$$

10) $\frac{dy}{dx} = x^2(y-1), f(0) = 3$, find y.

$\frac{dy}{y-1} = x^2dx \qquad \int \frac{dy}{y-1} = \int x^2dx \quad ln|y-1| = \frac{x^3}{3} + C$

$f(0) = 3 \quad ln2 = C \qquad ln|y-1| = \frac{x^3}{3} + ln2$

$|y-1| = e^{\frac{x^3}{3}+ln2} \qquad |y-1| = 2e^{\frac{x^3}{3}}$

$y - 1 = 2e^{\frac{x^3}{3}} \qquad OR \quad y - 1 = -2e^{\frac{x^3}{3}}$

$y = 1 + 2e^{\frac{x^3}{3}} \qquad OR \quad y = 1 - 2e^{\frac{x^3}{3}}$

$3 = 1 + 2e^0 = 3 \qquad 3 = 1 - 2e^0 = -1$

Yes No

$y = 1 + 2e^{\frac{x^3}{3}}$

11) $\frac{dy}{dx} = \frac{3-x}{y}, f(6) = -4$, find y.

$$ydy = (3 - x)dx \quad \int ydy = \int (3 - x)dx$$

$$\frac{y^2}{2} = 3x - \frac{x^2}{2} + C \qquad y^2 = 6x - x^2 + C \quad f(6) = -4$$

$$16 = 6(6) - 36 + C \quad 16 = C$$

$$y^2 = 6x - x^2 + 16 \qquad y = -\sqrt{6x - x^2 + 16}$$