9.4 Improper Integrals

$$\int_1^{\infty} \frac{1}{x^3} dx$$

You are taking the integral of an infinite region.

1) Express the improper integral as the limit of an integral.

$$\lim_{a\to\infty} \int_1^{a} \frac{1}{x^3} dx$$

2) Evaluate the integral by whatever method works.

$$\lim_{a\to\infty} \int_1^{a} \frac{1}{x^3} dx = \lim_{a\to\infty} \int_1^{a} x^{-3} dx = \lim_{a\to\infty} \frac{x^{-2}}{-2}\Bigg|_1^a = \lim_{a\to\infty} -\frac{1}{2x^2}\Bigg|_1^a$$

3) Evaluate the limit.

$$\lim_{a\to\infty} -\frac{1}{2x^2}\Bigg|_1^a = \lim_{a\to\infty} -\frac{1}{2a^2} + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

The area under an infinitely long curve is actually finite.

Since this area is finite, the integral converges to ½ .

Another example

$$\int_1^{\infty} \frac{1}{x} dx$$

1) Express the improper integral as the limit of an integral.

$$\lim_{a\to\infty} \int_1^{a} \frac{1}{x} dx = \lim_{a\to\infty} lnx\Big|_1^a =$$

2) Evaluate the integral by whatever method works.

$$\lim_{a\to\infty} lnx\Big|_1^{\infty}$$

3) Evaluate the limit.

$$\lim_{a\to\infty} (lna - ln1) = ln\infty$$

The improper integral cannot be evaluated, because the area it represents is infinite.

The integral diverges as it does not approach a value.

Can you create a pattern to know if the integral will converge or diverge?

Another Example

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

1) Express the improper integral as the limit of an integral.

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 x^{-\frac{1}{2}}dx =$$

2) Evaluate the integral by whatever method works.

$$\lim_{a\to 0^+} 2x^{\frac{1}{2}}\Big|_a^1$$

3) Evaluate the limit.

$$\lim_{a\to 0^+} 2\left(\sqrt{1}-\sqrt{a}\right) = 2(1-0) = 2$$

1985 36) $\int_{-1}^{1} \frac{3}{x^2}\,dx =$

$$\int_{-1}^{0} \frac{3}{x^2}\,dx + \int_0^1 \frac{3}{x^2}\,dx = lim_{b\to 0}(\int_{-1}^{b} \frac{3}{x^2}\,dx + \int_b^1 \frac{3}{x^2}\,dx) =$$

$$lim_{b\to 0} -\frac{3}{x}\Big|_{-1}^{b} + lim_{b\to 0} -\frac{3}{x}\Big|_{b}^{1} = lim_{b\to 0} -\frac{3}{b} - \left(-\frac{3}{-1}\right) + (-\frac{3}{1} - \left(-\frac{3}{b}\right))$$

$lim_{b\to 0} -\frac{3}{b} - 3 - 3 + \frac{3}{b} = undefined$ so the integral diverges.

1988 7) $\int_2^{+\infty} \frac{dx}{x^2} =$

$$lim_{b\to +\infty} \int_2^b \frac{dx}{x^2} = lim_{b\to +\infty} -\frac{1}{x}\Big|_2^b = lim_{b\to +\infty} -\frac{1}{b} + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

1993 11) $\int_4^{\infty} \frac{-2x}{\sqrt[3]{9-x^2}}\,dx =$

$$lim_{b\to\infty} \int_4^b \frac{-2x}{\sqrt[3]{9-x^2}}\,dx =$$

Use a u substitution. $u = 9 - x^2, du = -2xdx$

$$\frac{du}{dx} = -2x \qquad du = -2xdx$$

$$\int \frac{du}{\sqrt[3]{u}} = \int u^{-\frac{1}{3}}du = \frac{3}{2}u^{\frac{2}{3}} = lim_{b\to\infty} \frac{3}{2}(9-x^2)^{\frac{2}{3}}\Big|_4^b$$

$$lim_{b\to\infty} \frac{3}{2}(9-b^2)^{\frac{2}{3}} - lim_{b\to\infty} \frac{3}{2}(9-16)^{\frac{2}{3}} = nonexistant$$

1997 11) $\int_1^{\infty} \frac{x}{(1+x^2)^2} dx =$

$lim_{b\to\infty} \int_1^b \frac{x}{(1+x^2)^2} dx$

Use a u substitution. $u = 1 + x^2$

$\frac{du}{dx} = 2x \qquad \frac{du}{2} = xdx$

$\frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u} = lim_{b\to\infty} -\frac{1}{2(1+x^2)}\Big|_1^b = lim_{b\to\infty} -\frac{1}{(1+b^2)^2} + \frac{1}{(1+1^2)^2} = 0 + \frac{1}{4} = \frac{1}{4}$

1998 25) $\int_0^{\infty} x^2 e^{-x^3} dx = \quad u = -x^3 \quad du = -3x^2 dx \quad \frac{du}{-3} = x^2 dx$

$-\frac{1}{3}\int e^u du = -\frac{1}{3}e^u = lim_{b\to\infty} -\frac{1}{3}e^{-x^3}\Big|_0^b = lim_{b\to\infty} -\frac{1}{3e^{x^3}}\Big|_0^b =$

$lim_{b\to\infty} -\frac{1}{3e^{b^3}} + \frac{1}{3e^0} = \frac{1}{3}$

2008 11) Let R be the region between the graph of $y = e^{-2x}$ and the x-axis for $x \geq 3$. The area of R is

$\int_3^{\infty} e^{-2x} dx = lim_{b\to\infty} \int_3^b e^{-2x} dx = lim_{b\to\infty} -\frac{1}{2}e^{-2x}\Big|_3^b =$

$lim_{b\to\infty} -\frac{1}{2e^{2b}} + \frac{1}{2e^6} = \frac{1}{2e^6}$

1985 # 5

Let f be the function defined by $f(x) = -lnx$ for $0 < x \leq 1$ and let R be the region between the graph of f and the x-axis.

a) Determine whether the region R has finite area. Justify your answer.

$\int_0^1 -lnxdx = lim_{b\to 0} \int_b^1 -lnxdx$ Voodoo

$u = lnx \quad dv = 1$

$du = \frac{1}{x}dx \quad v = x$

$lim_{b\to 0} - (xlnx - \int_b^1 \frac{1}{x}(x)dx) = lim_{b\to 0} - (xlnx - x)|_b^1 =$

$lim_{b\to 0} - (1ln1 - 1) + (blnb - b) =$

$lim_{b\to 0} 1 + blnb - b = 1 + lim_{b\to 0} blnb - b = 1 + lim_{b\to 0} \frac{lnb}{\frac{1}{b}} - \frac{1}{\frac{1}{b}} =$

$1 + lim_{b\to 0} \frac{lnb - 1}{\frac{1}{b}}$ Use L'hopital's Rule

$1 + lim_{b\to 0} \frac{\frac{1}{b}}{-\frac{1}{b^2}} = 1 + lim_{b\to 0} - b = 1 - 0 = 1$

Determine whether the solid generated by revolving region R about the $\underline{y - axis}$

has finite volume.

When you rotate it about the y-axis, you change it to $x = something$.

Use a pancake formula. $A = \pi r^2$

$y = -lnx \quad -y = lnx \quad x = e^{-y}$

$\pi \int_0^\infty (e^{-y})^2 dy = \pi lim_{b\to\infty} - \frac{1}{2}e^{-2y}\Big|_0^b = \frac{1}{2}\pi lim_{b\to\infty}(-\frac{1}{e^b} + \frac{1}{e^0}) = \frac{\pi}{2}$

2001 # 5

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $lim_{x\to\infty} f(x) = 0$.

a) Evaluate $\int_1^{\infty} -3xf(x)dx$. Show the work that leads to your answer.

$$lim_{b\to\infty} \int_1^b -3xf(x)dx = lim_{b\to\infty} f(x)|_1^b = 0 - f(1) = -4$$

b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

Point	$f'(x) = -3xf(x)$	$\Delta x = .5$	dy	New Point
$(1, 4)$	$-3(1)(4) = -12$	$.5$	-6	$(1.5, -2)$
$(1.5, -2)$	$-3(1.5)(-2) = 9$	.5	4.5	(2, 2.5)

c) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

$$\int \frac{dy}{y} = \int -3xdx$$

$$ln|y| = -\frac{3x^2}{2} + C$$

$$y = e^{-\frac{3x^2}{2} + C} = e^{-\frac{3x^2}{2}} e^C \quad \text{Let } e^C = K$$

$$y = Ke^{-\frac{3x^2}{2}}$$

$$4 = Ke^{-\frac{3}{2}} \qquad \frac{4}{e^{-\frac{3}{2}}} = K \quad K = 4e^{\frac{3}{2}}$$

$$y = 4e^{\frac{3}{2}}e^{-\frac{3x^2}{2}}$$

2004 # 5

Let g be the function given by $g(x) = \frac{1}{\sqrt{x}}$.

a) Find the average value of g on the closed interval [1, 4].

The average value formula is $\frac{1}{b-a}\int_a^b f(x)dx$.

$$\frac{1}{4-1}\int_1^4 \frac{1}{\sqrt{x}}dx = \frac{1}{3}\int_1^4 x^{-\frac{1}{2}}dx = \frac{1}{3}(2)\, x^{\frac{1}{2}}\Big|_1^4 = \frac{2}{3}(2) - \frac{2}{3}(1) = \frac{2}{3}$$

b) Let S be the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis. Find the volume of S.

Use a pancake formula. $A = \pi r^2$

$\pi \int_1^4 (\frac{1}{\sqrt{x}})^2 dx = \pi ln x|_1^4 = \pi ln4 - \pi ln1 = \pi ln4$

c) For the solid S, given in part (b), find the average value of the areas of the cross sections perpendicular to the x-axis.

$$A = \pi r^2 = \pi(\frac{1}{\sqrt{x}})^2 = \frac{\pi}{x}$$

$\frac{1}{3}\int_1^4 \frac{\pi}{x}dx = \frac{\pi}{3} ln x|_1^4 = \frac{\pi}{3} ln4 - \frac{\pi}{3} ln1 = \frac{\pi}{3} ln4$

d) The average value of a function f on the unbounded interval $[a, \infty]$ is defined to be

$lim_{b\to\infty}\left[\frac{\int_a^b f(x)dx}{b-a}\right]$. Show that the improper integral $\int_4^\infty g(x)dx$ is divergent, but the average value of g on the interval $[4, \infty]$ is finite.

$\int_4^\infty \frac{1}{\sqrt{x}}dx = lim_{b\to\infty}\int_4^b (x)^{-\frac{1}{2}}dx = lim_{b\to\infty} 2(\sqrt{x})|_4^b = lim_{b\to\infty} 2\sqrt{b} - 4$ which diverges

$lim_{b\to\infty}\left[\frac{\int_a^b f(x)dx}{b-a}\right] = lim_{b\to\infty}\frac{1}{b-4}\int_4^b (x)^{-\frac{1}{2}}dx = lim_{b\to\infty}\frac{1}{b-4} 2(\sqrt{x})|_4^b =$

$lim_{b\to\infty}\frac{2\sqrt{b}-2\sqrt{4}}{b-4} = lim_{b\to\infty}\frac{2\sqrt{b}-4}{b-4} = 0$

2010 # 5

Let f and g be the functions defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{4x}{1+4x^2}$, for all $x > 0$.

a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

To find a maximum, find $g'(x)$, set it equal to 0, create a chart …..

$$g(x) = \frac{4x}{1+4x^2} \quad g'(x) = \frac{4(1+4x^2) - 4x(8x)}{(1+4x^2)^2} = \frac{4+16x^2-32x^2}{(1+4x^2)^2}$$

$$g'(x) = \frac{4-16x^2}{(1+4x^2)^2}$$

$g'(x) = 0$ when $4 - 16x^2 = 0 \quad 4 = 16x^2 \quad \frac{1}{4} = x^2 \quad x = \pm\frac{1}{2}$

The problem says on the open interval $(0, \infty)$ so we use $x = \frac{1}{2}$.

x	0	$\frac{1}{2}$	1
$f'(x)$	+	0	-

Since $g'(x) > 0$ for $0 < x < \frac{1}{2}$ and $g'(x) > 0$ for $x > \frac{1}{2}$, there is a maximum at $x = \frac{1}{2}$.

The value is $g\left(\frac{1}{2}\right) = \frac{4(\frac{1}{2})}{1+4(\frac{1}{2})^2} = \frac{2}{1+1} = 1$.

g has a maximum value of 1 at $x = \frac{1}{2}$ and g has no minimum value on the open interval $(0, \infty)$.

b) Find the area of the unbounded region in the first quadrant to the right of the vertical line $x = 1$, below the graph of f, and above the graph of g.

$$\int_1^{\infty} \big(f(x) - g(x)\big)dx = \int_1^{\infty} (\frac{1}{x} - \frac{4x}{1+4x^2})dx = lim_{b\to\infty}(lnx - \frac{1}{2}\ln(1+4x^2))|_1^b$$

$$lim_{b\to\infty}\left(lnb - \frac{1}{2}\ln(1+4b^2) - ln1 + \frac{1}{2}ln5\right) =$$

$$lim_{b\to\infty} \ln\left(\frac{b\sqrt{5}}{\sqrt{1+4b^2}}\right) = lim_{b\to\infty} \ln\left(\frac{\sqrt{5b^2}}{\sqrt{1+4b^2}}\right) = \frac{1}{2} lim_{b\to\infty} ln\left(\frac{5b^2}{1+4b^2}\right) = \frac{1}{2} ln\frac{5}{4}$$