8.2 L'Hopital's Rule

A whack function is defined as one of several weird functions:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty \cdot 0,\ \infty - \infty,\ 1^{\infty},\ 0^{0}\ \&\ \infty^{0}$$

If you have a function where you have limit that creates a whack function, you can use a special rule.

If $\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{0}{0}$ then this is true: $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$

And if you get lucky and $\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \frac{0}{0}$, you can do this:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{f''(x)}{g''(x)}$$

You are not doing a quotient rule, you are taking the derivative of the top and the derivative of the bottom.

Examples:

$$\lim_{x\to 0}\frac{sinx}{x} = \lim_{x\to 0}\frac{cosx}{1} = cos0 = 1$$

$$\lim_{x\to 0^-}\frac{sinx}{x^2} = \lim_{x\to 0^-}\frac{cosx}{2x} = \frac{-1}{0} = -\infty$$

This also works if $\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}$.

Example:

$$\lim_{x\to\infty}\frac{lnx}{2\sqrt{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \frac{1}{\sqrt{x}} = 0$$

Sometimes we need to change $\infty \cdot 0$ to $\frac{0}{0}$ by doing some algebra.

Examples:

$\lim_{x\to\infty} xsin\frac{1}{x}$

Let $h = \frac{1}{x}, so\ x = \frac{1}{h}$ $\quad \lim_{h\to 0^+}\frac{1}{h}sinh = \lim_{h\to 0^+}\frac{sinh}{h} = \lim_{h\to 0^+}\frac{cosh}{1} = 1$

Quick reminder of finding the derivatives with functions with functions as powers.

$y = x^x \qquad lny = lnx^x \qquad lny = xlnx$

$\frac{y'}{y} = lnx + x \cdot \frac{1}{x} \qquad \frac{y'}{y} = lnx + 1 \qquad y' = y(lnx + 1) = x^x(lnx + 1)$

You need to take the natural log of both sides.

Example:

$lim_{x\to 0^+}(\,x^x)$

Let $\lim_{x\to 0^+} y = lim_{x\to 0^+}(\,x^x)$

$\lim_{x\to 0^+} lny = (lim_{x\to 0^+} ln\,x^x)$

$= \lim_{x\to 0^+} xlnx$

$= \lim_{x\to 0^+} \frac{lnx}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0$

Remember that we found the $\lim_{x\to 0^+} lny = 0$.

If $lny = 0, y = 1$.

$lim_{x\to 0^+}(\,x^x) = 1$

This looks ugly, but stay calm.

$\lim_{x\to 1} \frac{\int_1^x \frac{dt}{t}}{x^3-1} = \frac{0}{0}$ so we can do L'Hopital's.

$\lim_{x\to 1} \frac{\int_1^x \frac{dt}{t}}{x^3-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{3x^2} = \frac{1}{3}$

1969 What is $lim_{x\to 0} \frac{e^{2x}-1}{tanx}$? $\quad lim_{x\to 0} \frac{2e^{2x}}{sec^2 x} = \frac{2}{1} = 2$

1973 $lim_{h\to 0}\frac{1}{h}\ln(\frac{2+h}{h})$ is $lim_{x\to 0}\frac{\ln(\frac{2+h}{h})}{h} = lim_{x\to 0}\frac{\frac{\frac{h-(2-h)}{h^2}}{\frac{2+h}{h}}}{h} = lim_{x\to 0}\frac{\frac{\frac{-2}{h^2}}{\frac{2+h}{h}}}{h}$

$$lim_{x\to 0}\frac{\frac{-2}{h^2}}{\frac{h(2+h)}{h}} = lim_{x\to 0}\frac{-2}{h^2}\cdot\frac{h(2+h)}{h} = lim_{x\to 0}\frac{-4-2h}{h^2} = \infty$$

1985 $lim_{x\to 0}\frac{1-cos^2(2x)}{x^2} = lim_{x\to 0}\frac{-2\cos(2x)(-\sin(2x))2}{2x} = lim_{x\to 0}\frac{4cos2xsin2x}{2x}$

$$lim_{x\to 0}\frac{4((cos2x)(2cos2x)+(-2sin2x)(sin2x))}{2} = \frac{4-0}{2} = 2$$

1985 $lim_{n\to\infty}\frac{4n^2}{n^2+10{,}000n} = 4$

1985 $lim_{x\to 0}(xcscx) = lim_{x\to 0}\frac{x}{sinx} = lim_{x\to 0}\frac{1}{cosx} = \frac{1}{1} = 1$

1985 $lim_{x\to\frac{\pi}{4}}\frac{\sin(x-\frac{\pi}{4})}{x-\frac{\pi}{4}} = lim_{x\to\frac{\pi}{4}}\frac{\cos(x-\frac{\pi}{4})}{1} = \frac{1}{1}=1$

1985 $lim_{x\to\infty}(1+5e^x)^{\frac{1}{x}} =$

$$y = (1+5e^x)^{\frac{1}{x}}$$

$$lny = ln(1+5e^x)^{\frac{1}{x}}$$

$$lny = \frac{1}{x}ln(1+5e^x)$$

$$lim_{x\to\infty}lny = lim_{x\to\infty}\frac{\ln(1+5e^x)}{x} = lim_{x\to\infty}\frac{\frac{5e^x}{1+5e^x}}{1} = lim_{x\to\infty}\frac{5e^x}{1+5e^x} = 1$$

$lim_{x\to\infty}lny = 1 \quad so\ lny = 1 \quad y = e$

1988 If $f'(x) = cosx$ and $g'(x) = 1$ for all x, and if $f(0) = g(0) = 0$, then

$$lim_{x\to 0}\frac{f(x)}{g(x)} = lim_{x\to 0}\frac{f\prime(x)}{g\prime(x)}$$

1993 If k is a positive integer, then $lim_{x\to +\infty}\frac{e^k}{e^x}$ is

A) 0 B) 1 C) e D) $k!$ E) nonexistent

$$lim_{x\to +\infty}\frac{e^k}{e^x} = lim_{x\to +\infty}\frac{1}{e^x} = 0$$

1993 $lim_{n\to\infty}\frac{3n^3-5n}{n^3-2n^2+1} = 3$

1993 $lim_{x\to 0}\frac{1-cosx}{2sin^2x} = lim_{x\to 0}\frac{-sinx}{4sinxcosx} = lim_{x\to 0}\frac{-1}{4cosx} = -\frac{1}{4}$

1993 Let f and g be functions that are differentiable for all real numbers, with $g(x) \neq 0$ for $x \neq 0$. If $lim_{x\to 0}f(x) = lim_{x\to 0}g(x) = 0$ and $lim_{x\to 0}\frac{f'(x)}{g'(x)}$ exists, then $lim_{x\to 0}\frac{f(x)}{g(x)}$ is $lim_{x\to 0}\frac{f'(x)}{g'(x)}$.

1997 $lim_{h\to 0}\frac{e^h-1}{2h} = lim_{h\to 0}\frac{e^h}{2} = \frac{1}{2}$

1998 $lim_{x\to 1}\frac{\int_1^x e^{t^2}dt}{x^2-1} = lim_{x\to 1}\frac{e^{x^2}}{2x} = \frac{e}{2}$

2003 $lim_{x\to 0}\frac{e^x - cosx - 2x}{x^2 - 2x} = lim_{x\to 0}\frac{e^x + sinx - 2}{2x - 2} = \frac{-1}{-2} = \frac{1}{2}$

2008 $lim_{x\to 0}\frac{sinxcosx}{x} = lim_{x\to 0}\frac{cosxcosx - sinxsinx}{1} = 1 - 0 = 1$

2010 # 5

Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$be the particular solution to this differential equation with the initial condition $f(1) = 0$.

For this particular solution, $f(x) < 1$ for all values of x.

a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

Original	$\frac{dy}{dx}$	dx	dy	New point
$(1, 0)$	1	-.5	-.5	$(.5, -.5)$
$(.5, -.5)$	1.5	-.5	-.75	$(0, -1.25)$

b) Find $lim_{x\to 1}\frac{f(x)}{x^3-1}$. Show the work that leads to your answer.

$$lim_{x\to 1}\frac{f(x)}{x^3-1} = lim_{x\to 1}\frac{f'(x)}{3x^2} = lim_{x\to 1}\frac{\frac{dy}{dx}}{3x^2} = lim_{x\to 1}\frac{1-y}{3x^2}$$
$$= lim_{x\to 1}\frac{1-0}{3x^2} = \frac{1}{3}$$

c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

$$\frac{dy}{dx} = 1 - y \qquad \frac{dy}{1-y} = dx \qquad -ln|1-y| = x + C$$

Use $(1,0)$ to solve for C. $\quad -\ln|0-1| = 1 + C \qquad 0 = 1 + C \quad C = -1$

$$-ln|1-y| = x - 1 \qquad ln|1-y| = 1 - x \qquad |1-y| = e^{1-x}$$

$Do\ we\ use\ the\ positive\ or\ negative\ case\ of\ the\ absolute\ value$?

Plug in the point given (1, 0) to work, so we use the positive case.

$$(1-y) = e^{1-x} \qquad -y = -1 + e^{1-x} \qquad y = 1 - e^{1-x}$$