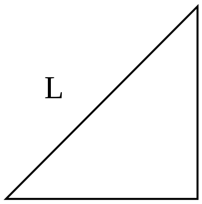
8.4 Length of a Curve

Every curve can be look at as a line if you look at just a small piece of it.

To find the length of the curve we will use the Pythagorean Theorem.



L dy

dx

$$L^2 = (dx)^2 + (dy)^2$$

$$L = \sqrt{(dx)^2 + (dy)^2}$$

$$\frac{L}{dx} = \sqrt{\frac{(dx)^2}{(dx)^2} + \frac{(dy)^2}{(dx)^2}} = \sqrt{1 + (\frac{dy}{dx})^2}$$

So the length of a curve form an x coordinate of $a\ to\ b$ is

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

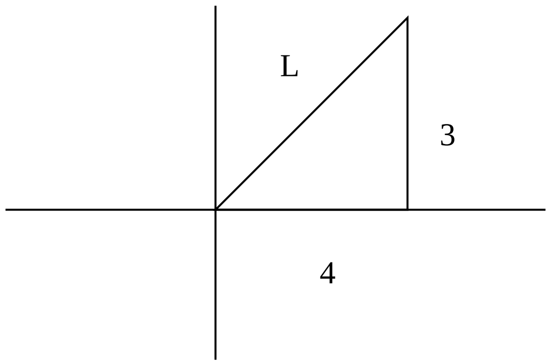
You do not need to be able to prove this formula.

If the equation of the curve is in the form of $x = \cdots$, you change the formula to

$$L = \int_a^b \sqrt{1 + (\frac{dx}{dy})^2}\, dy$$

Let’s show that this works.

Look at the line $y = \frac{3}{4}x$ from $x = 0\ to\ x = 4$.



The length of the curve from x = 0 to x = 4 is 5 by using the Pythagorean Theorem.

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_0^4 \sqrt{1 + (\frac{3}{4})^2} dx = \int_0^4 \sqrt{\frac{25}{16}}\, dx = \int_0^4 \frac{5}{4} dx = \frac{5}{4}x|_0^4 = 5 - 0 = 5$$

Let’s show it again using a semicircle.

The circumference of a semicircle is $C = \frac{1}{2}(2\pi r) = \pi r$ if the radius is $2, C = 2\pi$.

The equation of a circle is with radius 2 is $x^2 + y^2 = 4$.

The top half of the circle can be written as $y = \sqrt{4 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(-2x)\left(\frac{1}{\sqrt{4-x^2}}\right) = \frac{-x}{\sqrt{4-x^2}} \qquad (\frac{dy}{dx})^2 = \frac{x^2}{4-x^2}$$

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_{-2}^2 \sqrt{1 + \frac{x^2}{4-x^2}}\, dx = \int_{-2}^2 \sqrt{\frac{4-x^2+x^2}{4-x^2}}\, dx = \int_{-2}^2 \sqrt{\frac{4}{4-x^2}}\, dx =$$

$$\int_{-2}^2 \sqrt{\frac{1}{1-\frac{x^2}{4}}}\, dx = \int_{-2}^2 \sqrt{\frac{1}{1-(\frac{X}{2})^2}}\, dx = 2sin^{-1}(\frac{x}{2})|_{-2}^{2} = 2sin^{-1}1 - 2sin^{-1}(-1) =$$

$$2sin^{-1}1 - 2sin^{-1}(-1) = 2\left(\frac{\pi}{2}\right) - 2\left(-\frac{\pi}{2}\right) = \pi + \pi = 2\pi$$

1973 BC 10

The length of the curve $y = lnsecx$ from $x = 0\ to\ x = b$, may be represented by what integral?

$$\int_0^b \sqrt{1 + \left(\frac{secxtanx}{secx}\right)^2}\, dx = \int_0^b \sqrt{1 + tan^2dx}dx = \int_0^b secxdx$$

1985 BC 41

What is the length of the arc of $y = \frac{2}{3}x^{\frac{3}{2}}$ from $x = 0\ to\ x = 3$?

$$\int_0^3 \sqrt{1 + (\tfrac{3}{2} \cdot \tfrac{2}{3}x^{\frac{1}{2}})^2}dx = \int_0^3 \sqrt{1 + x}dx = (1 + x)^{\frac{3}{2}}\Big|_0^3 = (1 + 3)^{\frac{3}{2}} - 1 = 7$$

1988 BC 33

The length of the curve $y = x^3$ from $x = 0\ to\ x = 2$ is given by what integral?

$$\int_0^2 \sqrt{1 + (3x^2)^2}dx$$

2003 BC 15

The length of a curve from $x = 1\ to\ x = 4$ is given by $\int_1^4 \sqrt{1 + 9x^4}dx$. If the curve contains the point $(1, 6)$, which of the following could be an equation for this curve?

A) $y = 3 + 3x^2$ B) $y = 5 + x^3$ C) $y = 6 + x^3$

D) $y = 6 - x^3$ E) $y = \frac{16}{5} + x + \frac{9}{5}x^5$

$(\frac{dy}{dx})^2 = 9x^4$ $\frac{dy}{dx} = 3x^2$

$y = x^3 + C$ using $(1, 6)$ to solve C

$6 = 1 + C$ $C = 5$ The answer is B.

1973 BC3 C

Let R be the region enclosed by the graphs of $y = \ln(x^2 + 1)$ and $y = cosx$.

a) Find the area of R.

b) Write an expression involving one or more integrals that gives the length of the boundary of the region R. Do not evaluate.

c) The base of a solid is the region R. Each cross section of the solid perpendicular to the $x - axis$ is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

2008 Question 4

Let f be the function given by $f(x) = kx^2 - x^3$, where k is a positive constant. Let R be the region in the first quadrant bounded by the graph of f and the $x - axis$.

a) Find all values of the constant k for which the area of R is 2.

b) For $k > 0$, write, but do not evaluate , an integral expression in terms of k for the volume of the solid R is rotated about the $x - axis$.

c) For $k > 0$, write, but do not evaluate, an expression in terms of k, involving one or more integrals, that gives the perimeter of R.

2011 Question 3

Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of f, the coordinate axes, and the vertical line $x = k,$ where $k > 0$.

a) Write, but do not evaluate, an expression involving an integral that gives the perimeter of R in terms of k.

b) The region R is rotated about the $x - axis$ to form a solid. Find the volume, $V,$ of the solid in terms of k.

c) The volume V, found in part (b), changes as k changes. If $\frac{dk}{dt} = \frac{1}{3}$. Determine $\frac{dV}{dt}$ when $k = \frac{1}{2}$.

We will only find the length of the curve if it is considered smooth.

Smooth means that the function has a continuous first derivative.

Most of the time we will not be able to integrate this formula and we will have to use the calculator.

Here is one time that you can compute the integral by hand.

$$x = \frac{y^3}{3} + \frac{1}{4y} \quad from\ y = 1\ to\ y = 3$$

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2} \qquad \int_1^3 \left(\sqrt{1 + (y^2 - \frac{1}{4y^2})^2}\right) dy =$$

$$\int_1^3 \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}}\,dy = \int_1^3 \sqrt{y^4 + \frac{1}{2} + \frac{1}{16y^4}}\,dy = \int_1^3 \sqrt{(y^2 + \frac{1}{4y^2})^2}\,dy$$

$$\int_1^3 \left(y^2 + \frac{1}{4y^2}\right) dy = \frac{y^3}{3} - \frac{1}{4y}\Big|_1^3 = \frac{3^3}{3} - \frac{1}{4(3)} - \left(\frac{1}{3} - \frac{1}{4}\right) = 9 - \frac{1}{12} - \frac{1}{12} = 8\frac{10}{12} = 8\frac{5}{6}$$

9.1 Sequences

Defined explicitly means that the sequence is defined in terms of n. You can find any term with only the n value.

$$a_n = \frac{n}{5n-1}$$

Give the first 5 terms of this sequence:

n	1	2	3	4	5
$\frac{n}{5n-1}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{3}{14}$	$\frac{4}{19}$	$\frac{5}{24}$

Defined recursively: the sequence is defined in terms of previous terms. You can only find the next term.

$$a_n = 3a_{n-1}, a_1 = 2$$

Give the first 5 terms of this sequence:

n	1	2	3	4	5
$\frac{n}{5n-1}$	2	6	18	54	162

An arithmetic sequence is where the numbers go up or down by adding or subtracting a number.

Example:

1, 4, 7, 10, 13, ...

If a is your first term and you are increasing by a value of d, write a general formula for the sequence recursively.

$$a_n = a_{n-1} + 3, a_1 = 1$$

It can also be written as explicitly as:

$a_n = a_1 + (n-1)d$ where d is the common difference

A geometric sequence is where the numbers are increasing or decreasing by a factor of the number right before it.

$1, 2, 4, 8, 16, \dots \dots$

This can be written recursively as $a_n = ra_{n-1}$where r is the factor being multiplied.

This can also be written explicitly as $a_n = a_1 r^{n-1}$.

Does the sequence approach any specific value?

Let's look at some arithmetic sequences:

$d = 3$ $1, 4, 7, 10, 13, \dots ..$

$d = -6$ $1, -5, -11, -17, -23 \dots \dots$

Arithmetic sequences do not approach a specific value.

Let's look at some geometric sequences:

$r = 2$ $1, 2, 4, 8, 16 \dots \dots$

$r = -1$ $1, -1, 1, -1, 1 \dots \dots$

$r = \frac{1}{2}$ $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots \dots ..$

$r = -\frac{1}{3}$ $9, -3, 1, -\frac{1}{3}, \frac{1}{9} \dots ....$

If $-1 < r < 1$, the sequence approaches a certain value.

If a sequence approaches a certain value, it converges to that value.

If a sequence does not approach a certain value, it diverges.

The best way to see if the sequence converges or diverges is to find $\lim_{n\to\infty} a_n$. This is Bill Gates.

Determine the convergence or divergence of the sequence with the given nth term. If the sequence converges, find the limit.

$a_n = \frac{2n}{n+3}$ $\lim_{n\to\infty} a_n = \frac{2n}{n} = 2$

$a_n = (-1)^n \frac{n+1}{n^2+2}$ $\lim_{n\to\infty} (-1)^n \frac{1}{n} = 0$

$a_n = (1.1)^n$ This diverges as $r = 1.1$ which is outside of $(-1, 1)$.

$a_n = \cos(\frac{n\pi}{2})$ This diverges because you write it out, it flip flops from -1 to 0 to 1 to 0.

The Sandwich Theorem for Sequences

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

$a_n \le b_n \le c_n$ for all n > N, then $\lim_{n\to\infty} b_n = L.$

Show that the sequence $\{\frac{\cos n}{n}\}$ converges.

$$\left|\frac{\cos n}{n}\right| \le \frac{|\cos n|}{|n|} \le \frac{1}{n} \qquad \frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0 \; because \; \lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

If the limit exists, the sequence converges.

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0.$

8.2 L'Hopital's Rule

A whack function is defined as one of several weird functions:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty \cdot 0,\ \infty - \infty,\ 1^{\infty},\ 0^{0}\ \&\ \infty^{0}$$

If you have a function where you have limit that creates a whack function, you can use a special rule.

If $\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{0}{0}$ then this is true: $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f\prime(x)}{g\prime(x)}$

And if you get lucky and $\lim_{x\to 0}\frac{f\prime(x)}{g\prime(x)} = \frac{0}{0}$, you can do this:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f\prime(x)}{g\prime(x)} = \lim_{x\to 0}\frac{f\prime\prime(x)}{g\prime\prime(x)}$$

You are not doing a quotient rule, you are taking the derivative of the top and the derivative of the bottom.

Examples:

$$\lim_{x\to 0}\frac{sinx}{x} = \lim_{x\to 0}\frac{cosx}{1} = cos0 = 1$$

$$\lim_{x\to 0^-}\frac{sinx}{x^2} = \lim_{x\to 0^-}\frac{cosx}{2x} = \frac{-1}{0} = -\infty$$

This also works if $\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}$.

Example:

$$\lim_{x\to\infty}\frac{lnx}{2\sqrt{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \frac{1}{\sqrt{x}} = 0$$

Sometimes we need to change $\infty \cdot 0$ to $\frac{0}{0}$ by doing some algebra.

Examples:

$\lim_{x\to\infty} xsin\frac{1}{x}$

Let $h = \frac{1}{x}, so\ x = \frac{1}{h}$ $\quad \lim_{h\to 0^+}\frac{1}{h}sinh = \lim_{h\to 0^+}\frac{sinh}{h} = \lim_{h\to 0^+}\frac{cosh}{1} = 1$

Quick reminder of finding the derivatives with functions with functions as powers.

$y = x^x \qquad lny = lnx^x \qquad lny = xlnx$

$\frac{y'}{y} = lnx + x \cdot \frac{1}{x} \qquad \frac{y'}{y} = lnx + 1 \qquad y' = y(lnx + 1) = \ x^x(lnx + 1)$

You need to take the natural log of both sides.

Example:

$lim_{x \to 0^+}(\, x^x)$

Let $\lim_{x\to 0^+} y = lim_{x\to 0^+}(\, x^x)$

$\lim_{x\to 0^+} lny = (lim_{x\to 0^+} ln\, x^x)$

$= \lim_{x\to 0^+} xlnx$

$= \lim_{x\to 0^+} \frac{lnx}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0$

Remember that we found the $\lim_{x\to 0^+} lny = 0$.

If $lny = 0, y = 1$.

$lim_{x\to 0^+}(\, x^x) = 1$

This looks ugly, but stay calm.

$\lim_{x\to 1} \frac{\int_1^x \frac{dt}{t}}{x^3-1} = \frac{0}{0}$ so we can do L'Hopital's.

$\lim_{x\to 1} \frac{\int_1^x \frac{dt}{t}}{x^3-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{3x^2} = \frac{1}{3}$

9.3 Relative Rates of Growth

The whole objective of this section is to compare two functions and see which function grows at a quicker rate.

The formula used is $\lim_{n\to\infty} \frac{f(x)}{g(x)} = L.$

If $L = 0, g(x)$ grows faster.

If $L = \infty, f(x)$ grows faster.

If L is a finite number, $\neq 0, f(x) \& g(x)$ grow at the same rate.

Let's compare $x^3 - 3x + 1$ with e^x. It does not matter which function you put on the top or bottom.

$\lim_{n\to\infty} \frac{x^3-3x+1}{e^x}$ = use L'Hopital's Rule = $\lim_{n\to\infty} \frac{3x^2-3}{e^x} = \frac{6x}{e^x} = \frac{6}{e^x} = \frac{6}{huge} = 0$

Let's compare $f(x) = \sqrt{x}$ with $g(x) = \sqrt{10x+1}$

$$\lim_{n\to\infty} \frac{\sqrt{10x+1}}{\sqrt{x}} = \sqrt{\frac{10x+1}{x}} = \sqrt{\frac{10x}{x} + \frac{1}{x}} = \lim_{n\to\infty} \sqrt{10 + \frac{1}{x}} = \sqrt{10 + \frac{1}{huge}} = \sqrt{10} = L$$