New 8.3 Volume

We will find three basic types of volume with variations on each type.

They are pancakes, donuts and pop-ups.

Pancakes

Pancakes are when you rotate a shape about a vertical or horizontal line. The shape must attach to the line. There cannot be any space between the shape and the line. When you create a pancake, you are basically adding circles together. Usually the radius of the circle is changing. This can also be called a disk.

Rotate $y = x^2$ about the x-axis and find the volume from $x = 0 \ to \ x = 4$.

The area of a circle is $A = \pi r^2$ with this radius being x^2.

$$\int_0^4 \pi(x^2)^2 dx = \int_0^4 \pi x^4 dx = \frac{\pi x^5}{5}\Bigg|\begin{matrix}4\\0\end{matrix} = \frac{4^5}{5} - \frac{0^5}{5} = \frac{256\pi}{5}$$

Rotate $y = \sqrt[3]{x}$ about the x-axis and find the volume from $x = 1\ to\ x = 8$.

The area of a circle is $A = \pi r^2$ with this radius being $\sqrt[3]{x}$.

$$\int_1^8 \pi(\sqrt[3]{x})^2 dx = \int_1^8 \pi(x)^{\frac{2}{3}} dx = \frac{3\pi x^{\frac{5}{3}}}{5}\Bigg|_1^8 = \frac{3\pi}{5}(32) - \frac{3\pi}{5}(1) = \frac{92\pi}{5}$$

Rotate $y = x^2$ about the y-axis and find the volume from $x = 0\ to\ x = 4$.

If we are rotating it about the y-axis, we have to solve for x. $x = \sqrt{y}$

If the x values are going from 0 to 4, the y values are going from 0 to 2.

The area of a circle is $A = \pi r^2$ with this radius being $\sqrt{y}$.

$$\int_0^2 \pi(\sqrt{y})^2 dy = \int_0^2 \pi y \, dy = \frac{\pi y^2}{2}\Big|_0^2 = 2\pi$$

Donuts

Donuts are created when you are rotating a shape about a line but there is space between the line and the shape. This creates a hole between the two circles that are formed. The function farthest from the axis is the R and the function closest to the axis is the r. The area formed by a donut can be found using the formula $A = \pi(R^2 - r^2)$. This can also be called a washer.

Let T be the region formed by the line $y = 4x$ & $y = x$. Rotate T about the x-axis and find the $volume\ from\ x = 0\ to\ x = 4$.

$y = 4x$ is farthest from the x-axis and so $R = 4x$.

$y = x$ is closest to the axis, so this is my r.

$A = \pi(R^2 - r^2) = \pi((4x)^2 - (x)^2) = 15\pi x^2$

To find the volume, we integrate the area.

$\int_0^4 15\pi x^2 dx = \frac{15\pi x^3}{3} = 5\pi x^3 \Big|_0^4 = 320\pi$

Let T be the region formed by the line $y = 4$ & $y = x^2$. Rotate T about the x-axis and find the volume from $x = 0\ to\ x = 2$.

$y = 4$ is farthest from the x-axis and so $R = 4$.

$y = x^2$ is closest to the axis, so this is my r.

$A = \pi(R^2 - r^2) = \pi((4)^2 - (x^2)^2) = \pi(16 - x^4)$

To find the volume, we integrate the area.

$\int_0^2 \pi(16 - x^4)dx = 16\pi x - \frac{\pi x^5}{5}\Big|_0^2 = 64\pi - \frac{32\pi}{5} = \frac{288\pi}{5}$

Let T be the region formed by $y = x^2$ and $y = 4$ from $x = 0$ to $x = 2$. Rotate T about the line $x = 7$ and find the volume.

When you are rotating the figure about $x = 7$, you need to change all of functions to $x =$.

$$y = x^2 \rightarrow x = \sqrt{y}$$

$x = \sqrt{y}$ is farthest from $x = 7$ so $R = 7 - \sqrt{y}$

$x = 4$ is closest to $x = 7$ so $r = 7 - 4 = 3$

$$A = \pi(R^2 - r^2) = \pi((7 - y)^2 - (3)^2) = \pi(49 - 23y + y^2)$$

To find the volume, we integrate the area.

$$\int_0^2 \pi(49 - 23y + y^2)dy = 49\pi y - \frac{23\pi y^2}{2} + \frac{\pi y^3}{3}\Big|_0^2 = 98\pi - 46\pi + \frac{8\pi}{3} = \frac{164\pi}{3}$$

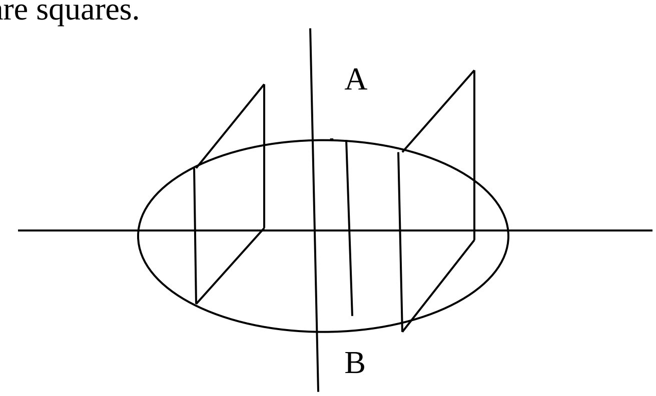
Pop-ups

A pop-up is when you have a base shape and then you create another shape that is perpendicular to the base. This is an example of a figure with the base is a circle and the perpendicular shapes are squares.

$$x^2 + y^2 = 4$$

$$y^2 = 4 - x^2$$

$$y = \sqrt{4 - x^2}$$



You find the area of the perpendicular shape, in this case a square.

$A = s^2$ The side is the diameter of the circle (AB). AB is also $2y = 2\sqrt{4 - x^2}$

$$A = (2\sqrt{4 - x^2})^2 = 4(4 - x^2) = 16 - 4x^2$$

If we integrate the area, we get the volume.

$$\int_{-2}^2 (16 - 4x^2)dx = 16x - \frac{4x^3}{3}\Big|_{-2}^2 = 16(2) - \frac{4(2)^3}{3} - \left(16(-2) - \frac{4(-2)^3}{3}\right) =$$

$$32 - \frac{32}{3} + 32 - \frac{32}{3} = 64 - \frac{64}{3} = \frac{128}{3}$$

Proving the formula for the volume of a sphere.

The perpendicular shapes are circles with Area -= πr^2

Your basic shape is a circle with formula $x^2 + y^2 = r^2$.

The radius of the perpendicular shape is the diameter of your circle

or $y = \sqrt{r^2 - x^2}$.

$A = \pi(\sqrt{r^2 - x^2})^2 = \pi(r^2 - x^2)$

We will integrate the areas to find the volume.

$\int_{-r}^{r}(r^2 - x^2)dx = (r^2x - \frac{x^3}{3})\Big|_{-r}^{r} = \left(r^2r - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{(-r^3)}{3}\right) =$

$\left(r^3 - \frac{r^3}{3}\right) + \left(r^3 - \frac{r^3}{3}\right) = \frac{2\pi r^3}{3} + \frac{2\pi r^3}{3} = \frac{4\pi r^3}{3}$