Calculus 7.5 Partial Fractions

$$\frac{1}{3}+\frac{5}{7}=\frac{3(5)+1(5)}{3(7)}=\frac{15+5}{21}=\frac{20}{21}$$

$$\frac{2}{x+5}+\frac{7}{x-2}=\frac{2(x-2)+7(x+5)}{(x+5)(x-2)}=\frac{2x-4+7x+35}{(x+5)(x-2)}=\frac{9x+31}{(x+5)(x-2)}=\frac{9x+31}{x^2+3x-10}$$

$$\frac{9x+31}{x^2+3x-10}=\frac{9x+31}{(x+5)(x-2)}=\frac{A}{x+5}+\frac{B}{x-2} = \frac{A(x-2)+B(x+5)}{(x+5)(x-2)}$$

$$9x+31=A(x-2)+B(x-5)$$

Let $x=2$ $\quad 9(2)+31=A(2-2)+B(2+5) \quad 49=7B \quad 7=B$

Let $x=-5$ $\quad 9(-5)+31=A(-5-2)+B(-5+5) \quad -14=-7A \quad 2=A$

$$\frac{9x+31}{x^2+3x-10}=\frac{2}{x+5}+\frac{7}{x-2}$$

The purpose of partial fractions is to change the form of the function so that it can be integrated.

If we saw $\int \frac{9x+31}{x^2+3x-10}dx$ we would write it as

$$\int \frac{2}{x+5}dx+\int \frac{7}{x-2}dx=2ln|x+5|+7ln|x-2|+C$$

$$\int \frac{5}{x^2-1}dx \qquad \frac{5}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{x^2-1}$$

$$5 = A(x+1) + B(x-1)$$

Let $x = -1$ $\quad 5 = A(-1+1) + B(-1-1) \quad 5 = -2B \quad -\frac{5}{2} = B$

Let $x = 1$ $\quad 5 = A(1+1) + B(1-1) \quad 5 = 2A \quad \frac{5}{2} = A$

$$\int \frac{5}{x^2-1}dx = \int \frac{\frac{5}{2}}{x-1}dx + \int \frac{-\frac{5}{2}}{x+1}dx = \frac{5}{2}\int \frac{1}{x-1}dx - \frac{5}{2}\int \frac{1}{x+1}dx = \frac{5}{2}ln|x-1| - \frac{5}{2}ln|x+1| + C$$

$$\frac{5}{2}ln|x-1| - \frac{5}{2}ln|x+1| + C = \frac{5}{2}ln\left|\frac{x-1}{x+1}\right| + C = ln\left|\frac{x-1}{x+1}\right|^{\frac{5}{2}} + C$$

$$\int \frac{3x+2}{(5x-3)(7-2x)}dx \qquad \frac{3x+2}{(5x-3)(7-2x)} = \frac{A}{5x-3} + \frac{B}{7-2x} = \frac{A(7-2x)+B(5x-3)}{(5x-3)(7-2x)}$$

$$3x+2 = A(7-2x) + B(5x-3)$$

Let $A = \frac{7}{2}$ $\quad 3\left(\frac{7}{2}\right) + 2 = A\left(7 - 2\left(\frac{7}{2}\right)\right) + B(5\left(\frac{7}{2}\right) - 3)$

$\frac{21}{2} + 2 = B\left(\frac{35}{2} - 3\right) \quad \frac{25}{2} = \frac{29}{2}B \quad B = \frac{25}{29}$

Let $B = \frac{3}{5}$ $\quad 3\left(\frac{3}{5}\right) + 2 = A\left(7 - 2\left(\frac{3}{5}\right)\right) + B(5\left(\frac{3}{5}\right) - 3)$

$\frac{9}{5} + 2 = A\left(7 - \frac{6}{5}\right) \quad \frac{19}{5} = \frac{29}{5}A \quad A = \frac{19}{29}$

$$\int \frac{3x+2}{(5x-3)(7-2x)} = \int \frac{\frac{19}{29}}{5x-3}dx + \int \frac{\frac{25}{29}}{7-2x}dx = \frac{19}{29}\int \frac{1}{5x-3}dx + \frac{25}{29}\int \frac{1}{7-2x}dx$$

$$\frac{19}{29}\int \frac{1}{5x-3}dx = \frac{19}{29}\cdot\frac{1}{5}ln|5x-3| + C = \frac{19}{145}ln|5x-3| + C$$

$$\frac{25}{29}\int \frac{1}{7-2x}dx = \frac{25}{29}\cdot\frac{-1}{2}ln|7-2x| + C = \frac{-25}{58}ln|7-2x| + C$$

$$\frac{19}{145}ln|5x-3| + \frac{-25}{58}ln|7-2x| + C$$

To combine this into division would be extremely ugly!!!

This is just for the people that think they want ugly. $\quad ln\left|\frac{(5x-3)^{\frac{19}{5}}}{(7-2x)^{\frac{25}{2}}}\right|^{\frac{1}{29}} + C$

Three factors

$$\int \frac{5}{x^3-4x}dx \qquad \frac{5}{x^3-4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}=\frac{A(x-2)(x+2)+Bx(x+2)+Cx(x-2)}{x(x-2)(x+2)}$$

$$5=A(x-2)(x+2)+Bx(x+2)+Cx(x-2)$$

Let $x=2$ $\quad 5=A(2-2)(2+2)+2B(2+2)+2C(2-2)$

$$5=8B \qquad \frac{5}{8}=B$$

Let $x=0$ $\quad 5=A(0-2)(0+2)+B(0)(0+2)+C(0)(0-2)$

$$5=-4A \qquad -\frac{5}{4}=A$$

Let $x=-2$ $\quad 5=A(-2-2)(-2+2)-2B(-2+2)-2C(-2-2)$

$$5=8C \qquad \frac{5}{8}=C$$

$$\int \frac{5}{x^3-4x}dx=\int \frac{-\frac{5}{4}}{x}dx+\int \frac{\frac{5}{8}}{x-2}dx+\int \frac{\frac{5}{8}}{x+2}dx=$$

$$-\frac{5}{4}\int \frac{dx}{x}+\frac{5}{8}\int \frac{dx}{x-2}+\frac{5}{8}\int \frac{dx}{x+2}=$$

$$-\frac{5}{4}ln|x|+\frac{5}{8}ln|x-2|+\frac{5}{8}ln|x+2|+C$$

$$ln\left|\frac{((x-2)(x+2))^{\frac{5}{8}}}{x^{\frac{5}{4}}}\right|+C=ln\left|\frac{((x-2)(x+2)}{x^2}\right|^{\frac{5}{8}}+C$$

Long Division

When the numerator has a larger power than the denominator, you need to do long division before using partial fractions or something else.

$$\int \frac{2x^3}{x^2 - 1} dx$$

$$\begin{array}{r l} & 2x \\ x^2 + 0x - 1 \;) & 2x^3 + 0x^2 + 0x + 0 \\ & \underline{2x^3 + 0x^2 - 2x} \\ & \qquad\qquad\quad 2x \end{array}$$

$$\frac{2x^3}{x^2-1} = 2x + \frac{2x}{x^2-1}$$

$$\int \frac{2x^3}{x^2-1} dx = \quad \int 2xdx + \int \frac{2x}{x^2-1} dx$$

$$x^2 + \int \frac{2x}{x^2-1} dx \quad \rightarrow \quad u = x^2 - 1$$

$$\frac{du}{dx} = 2x$$

$$du = 2xdx$$

$$x^2 + \int \frac{du}{u} = x^2 + ln|u| + C = x^2 + ln|x^2 - 1| + C$$

Repeated factors

$\int \frac{2x+9}{(x+1)^3}dx$ has to be broken down into $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$

$$\frac{2x+9}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$2x + 9 = A(x+1)^2 + B(x+1) + C$

Let $x = -1$ $\quad 2(-1) + 9 = A(-1+1)^2 + B(-1+1) + C$

$7 = C$

Let $x = 0$ $\quad 2(0) + 9 = A(0+1)^2 + B(0+1) + C$

$9 = A + B + C$

$9 = A + B + 7 \quad 2 = A + B$

Let $x = 1$ $\quad 2(1) + 9 = A(1+1)^2 + B(1+1) + C$

$11 = 4A + 2B + C$

$11 = 4A + 2B + 7 \quad 4 = 4A + 2B \quad 2 = 2A + B$

$2 = A \ + B \quad -2 = -A - B$

$2 = 2A + B \quad 2 = 2A + B$

$0 = A$

Go back to any equation you have with A, B & C

$9 = A + B + C \qquad A = 0, C = 7 \qquad 9 = 0 + B + 7 \quad B = 2$

$$\int \frac{2x+9}{(x+1)^3}dx = \int \frac{0}{x+1}dx + \int \frac{2}{(x+1)^2}dx + \int \frac{7}{(x+1)^3}dx$$

$$\int \frac{2x+9}{(x+1)^3}dx = \int \frac{2}{(x+1)^2}dx + \int \frac{7}{(x+1)^3}dx$$

$2\int (x+1)^{-2}dx + 7\int (x+1)^{-3}dx$

$-2(x+1)^{-1} - \frac{7}{2}(x+1)^{-2} + C$

$\frac{dP}{dt} = kP(M-P)$ is a logistic differential equation

The P is changing at a rate based on a factor k, the existing amount P and the difference between the carrying capacity M and the existing amount P.

$\frac{dP}{dt} = kP(M-P)$ changes to $P = \frac{M}{1+Ae^{-Mkt}}$ with a little work.

To do this takes a little work (22 steps) worth 10 Extra Credit points.

$$\frac{dP}{dt} = kP(M-P) \qquad \frac{dP}{P(M-P)} = kdt \qquad \int \frac{dP}{P(M-P)} = \int kdt$$

$\int \frac{dP}{P(M-P)}$ uses partial fractions.

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P} \qquad \frac{A(M-P) + BP}{P(M-P)}$$

$$1 = A(M-P) + BP$$

Let $P = M$ $\quad 1 = A(M-M) + BM \qquad 1 = BM \qquad B = \frac{1}{M}$

Let $P = 0$ $\quad 1 = A(M-0) + B(0) \qquad 1 = AM \qquad A = \frac{1}{M}$

$$\int \frac{dP}{P(M-P)} = \int \frac{\frac{1}{M}}{P} dP + \int \frac{\frac{1}{M}}{M-P} dP = \frac{1}{M}\int \frac{1}{P} dP + \frac{1}{M}\int \frac{1}{M-P} dP$$

$$\frac{1}{M} ln|P| - \frac{1}{M} ln|M-P| = kt + C$$

Since P is positive and M – P is positive, we will eliminate the absolute values of the logarithm.

$$lnP - \ln(M-P) = mkt + C$$

$$ln\left(\frac{P}{M-P}\right) = mkt + C \qquad -ln\left(\frac{P}{M-P}\right) = -mkt + C \qquad ln\left(\frac{P}{M-P}\right)^{-1} = -mkt + C$$

$ln\left(\frac{M-P}{P}\right) = -mkt + C$ $\qquad$ Since $e^{\ln(something)} = something$

$$\frac{M-P}{P} = e^{-mkt+C} \qquad \frac{M}{P} - \frac{P}{P} = e^{-mkt+C} \qquad \frac{M}{P} = 1 + e^{-mkt+C}$$

$$\frac{P}{M} = \frac{1}{1+e^{-mkt+C}} \qquad P = \frac{M}{1+e^{-mkt+C}} \qquad P = \frac{M}{1+e^{C}e^{-mkt}} \quad let\ e^{c} = A \qquad P = \frac{M}{1+Ae^{-mkt}}$$

$\frac{dP}{dt} = kP(M - P)$ is equivalent to $P = \frac{M}{1+Ae^{-Mkt}}$

If $\frac{dP}{dt} = .006P(200 - P)$ and P = 8 when t = 0, solve for P.

$k = .006 \qquad M = 200$

$P = \frac{200}{1+Ae^{-200(.006)t}} = \frac{200}{1+Ae^{-1.2t}}$

$8 = \frac{200}{1+Ae^{-1.2(0)}} \qquad 8 = \frac{200}{1+A} \qquad 8(1 + A) = 200 \qquad 1 + A = 25 \qquad A = 24$

$P = \frac{200}{1+24e^{-1.2t}}$