New Calculus 5.4 Modeling and Optimization (Word Problems ALL DAY)

The sum of two non-negative numbers is 30.

a) Find the numbers if the sum of their squares is as large as possible.
b) Find the numbers if the sum of their squares is as small as possible.
c) Find the numbers if one number plus the square root of the other is as large as possible.
d) Find the numbers if one number plus the square root of the other is as small as possible.

Always play with one variable if you can.

Let $x =$ the first number and $30 - x$ = the second number. The boundaries of this problem are [0, 30].

a & b) The sum of the squares means $x^2 + (30 - x)^2$.

To make this as large as possible, create a function, $f(x)$ and take the derivative, set it equal to 0 and find the absolute maximum and minimum.

$y = x^2 + (30 - x)^2$

$y' = 2x + 2(30 - x)(-1) = 2x - 60 + 2x = 4x - 60$

$0 = 4x - 60$

$x = 15$

x	0	15	30
$f'(x)$	-	0	+
Local Max or min		Local Min	
$f(x)$	900	450	900
Absolute Max or min	Absolute Max	Absolute Min	Absolute Max

The absolute max occurs when the numbers are 0 & 30.

The absolute minimum occurs when the numbers are 15 & 15.

c & d) One number plus the square root of the other is as large as possible

$y = x + \sqrt{30 - x}$ $\quad y = 30 - x + \sqrt{x}$ We get to pick. Choose the second.

$y' = -1 + \frac{1}{2\sqrt{x}} = 0 \quad \frac{1}{2\sqrt{x}} = 1 \quad 2\sqrt{x} = 1 \quad \sqrt{x} = \frac{1}{2} \quad x = \frac{1}{4}$

x	0	$\frac{1}{9}$	¼	30
$f'(x)$	undefined	+	0	-
Local Max or min			Local Max	
$f(x)$	$\sqrt{30}$		$30\frac{1}{4}$	$\sqrt{30}$
Absolute Max or min	Absolute Min		Absolute Max	Absolute Min

The maximum values occur with the numbers $\frac{1}{4}$ & $29\frac{3}{4}$..

The minimum values occur with the number 0 & 30.

What is the smallest perimeter possible for a rectangle whose area is $16\ in^2$ and what are the dimensions?

The area of a rectangle is $A = bh$. If $A = 16$, let x be one side and the other side is $\frac{16}{x}$.

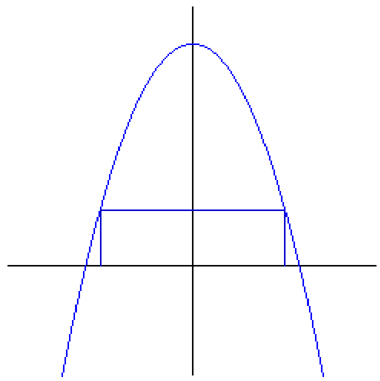
The perimeter of a rectangle is $P = 2b + 2h = 2x + 2\left(\frac{16}{x}\right) = 2x + \frac{32}{x}$

To find the smallest perimeter, we find the derivative ……

$P' = 2 - \frac{32}{x^2} = 0 \quad 2 = \frac{32}{x^2} \quad 2x^2 = 32 \quad x^2 = 16 \quad x = \pm 4 \to 4$

The dimensions are 4 x 4.

A rectangle has its base on the x-axis and its upper vertices on the parabola $y = 12 - x^2$.



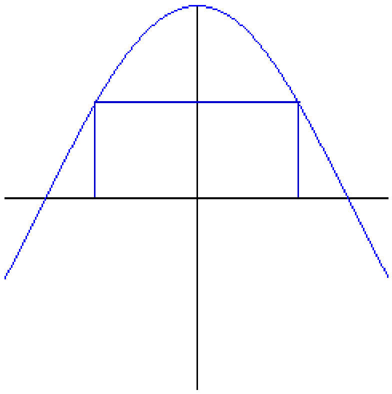
The area of a rectangle is $A = bh$.

$$A = 2x(12 - x^2) = 24x - 2x^3$$

$$A' = 24 - 6x^2 = 0 \quad 24 = 6x^2 \quad 4 = x^2 \quad x = \pm 2 \to 2$$

x	0	2	$\sqrt{12}$
$f'(x)$	+	0	-
Local Max or min		Local max	
$f(x)$	0	32	0
Absolute Max or min		Absolute max	

Find the largest area that can be inscribed in $y = cosx$ with the x-axis as its base.

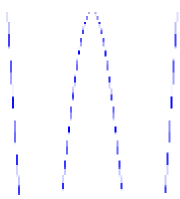


Area = $2x(cosx)$

$A' = 2cosx - 2xsinx = 0$

You need to use a calculator to find this answer. Calculus calculators always use radians.

Put in $y = 2cosx - 2xsinx$ and find the zeros.



Your graph will intersect the x-axis many times. You want the first zero to the right of the x axis (the smallest x value). That is x = .86.

x	0	$.86$	1.57 ($\frac{\pi}{2}$)
$f'(x)$	+		-
Local Max or min		Local max	
$f(x)$	0		0
Absolute Max or min		Absolute max	

The dimensions of the area are $A = 2(.86) by \cos(.86)\ or$

You are planning to make an open rectangular box from an 8 by 15 inch piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?

$$V = x(8-2x)(15-2x) = 120x - 46x^2 + 4x^3$$

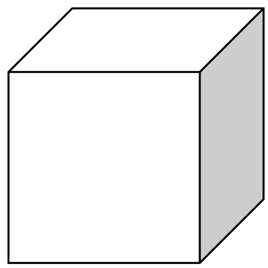
$$V' = 120 - 92x + 12x^2 = 4(30 - 23x + 3x^2) = 4(x-6)(3x-5)$$

$V' = 0\ at\ x = 6, \frac{5}{3} \rightarrow x \neq 6$ as the side length would be a negative number.

x	0	$\frac{5}{3}$	4
$f'(x)$	+	0	-
Local Max or min		Local max	
$f(x)$	0	$\frac{2450}{27}$	0
Absolute Max or min		Absolute max	

If $x = \frac{5}{3}$, the dimensions are $\frac{5}{3}\, by\, \frac{14}{3}\, by\, \frac{35}{3}$ and the volume is $\frac{2450}{27}$.

Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surface area is 337.5 square centimeters.



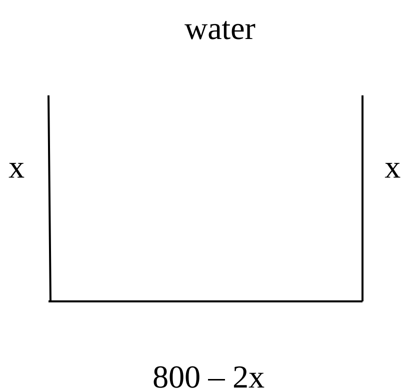
$SA = 2x^2 + 4xh \qquad 337.5 = 2x^2 + 4xh \qquad \frac{337.5 - 2x^2}{4x} = h$

$V = x^2h = x^2\left(\frac{337.5 - 2x^2}{4x}\right) = 84.375x - .5x^3$

$V' = 84.375 - 1.5x^2 = 0 \qquad 84.375 = 1.5x^2 \qquad 56.25 = x^2 \qquad x = 7.5$

The biggest volume would be a cube with side length = 7.5 .

A rectangular plot of farmland will be bounded by a river and on the other three sides by a single-strand electric fence. With 800 meters of wire at your disposal, what is the largest area that you can enclose, and what are its dimensions?



$A = x(800 - 2x)$

$A = 800x - 2x^2$

$A' = 800 - 4x$

$0 = 800 - 4x$

$800 = 4x$

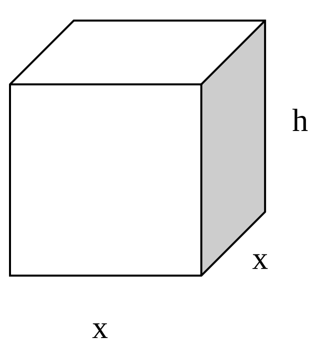
200 = x

x	0	200	400
$f'(x)$	+	0	-
Local Max or min		Local max	
$f(x)$	0	80,000	0
Absolute Max or min		Absolute max	

The dimensions are 200 m by 400 m and the volume is 80,000 m.

Your iron works has contracted to design and build a $500 - ft^3$, square-based, open-top, rectangular holding tank for a paper company. The tank is to be made by welding thin stainless plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.

What dimensions do you tell the shop to use?



$V = x^2 h \qquad 500 = x^2 h \qquad h = \frac{500}{x^2}$

You are finding the surface area of the rectangular box. You add all of the areas of the sides together. You have four walls with area = xh and one square with side x and area = x^2.

$$SA = 4xh + x^2 = 4x\left(\frac{500}{x^2}\right) + x^2 = \frac{2000x}{x^2} + x^2 = \frac{2000}{x} + x^2 = 2000x^{-1} + x^2$$

$SA = 2000x^{-1} + x^2$

$SA' = -2000x^{-2} + 2x$

$0 = -2000x^{-2} + 2x$ Multiply both sides by x^2 to get rid of the denominator.

$0 = -2000 + 2x^3$

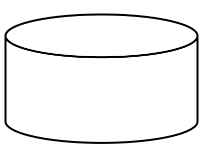
$2000 = 2x^3$

$1000 = x^3$

$10 = x$

x	1	10	20
$f'(x)$	-	0	+
Local Max or min		Local min	
$f(x)$	2001	300	500
Absolute Max or min		Absolute min	

What are the dimensions of the lightest open-top right cylindrical can that will hold a volume of 1000 cm^3?



$V = \pi r^2 h$

$1000 = \pi r^2 h$

$\frac{1000}{\pi r^2} = h$

$SA = 2\pi rh + \pi r^2$

$SA = 2\pi r\left(\frac{1000}{\pi r^2}\right) + \pi r^2 = \left(\frac{2000}{r}\right) + \pi r^2 = 2000r^{-1} + \pi r^2$

$SA' = -2000r^{-2} + 2\pi r = 0$=

$\frac{-2000}{r^2} + 2\pi r = 0$ Multiply by r^2.

$-2000 + 2\pi r^3 = 0$

$2\pi r^3 = 2000$

$r^3 = \frac{2000}{2\pi} = \frac{1000}{\pi}$

$r = \frac{10}{\sqrt[3]{\pi}}, h = \frac{1000}{\pi(\frac{10}{\sqrt[3]{\pi}})^2} = \frac{10}{\sqrt[3]{\pi}}$