10.5 Testing Convergence at Endpoints

Previous Tests for Convergence or Divergence

1) Arithmetic
2) Geometric
3) Ratio
4) Nth Term
5) Direct Comparison
6) Telescoping Series

New Tests

7) The Integral

If the integral diverges, the series diverges.

If the integral converges, the series converges.

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

Look at $\lim_{a\to\infty} \int_1^a \frac{lnx}{x}\,dx =$

$$\lim_{a\to\infty} \frac{(lnx)^2}{2}\Big|_1^a \lim_{a\to\infty} \frac{(lna)^2}{2} - \frac{(ln1)}{2} = \lim_{a\to\infty} \frac{(lna)^2}{2} \to \infty$$

The integral diverges so the series diverges.

8) The P-Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ If $p > 1$, the series converges.

If $p \leq 1$, the series diverges.

$\sum_{n=1}^{\infty} \frac{1}{n}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

$p = 1$ diverges $\quad p = \frac{1}{2}$ diverges $\quad \frac{1}{n^2}$ p = 2, converges

9) The Limit Comparison

Take one series ($\sum a_n$)that you know converges or diverges and another series that you don't know about ($\sum b_n$).

$\lim_{n\to\infty} \frac{a_n}{b_n} = a\ value$ both series converge or diverge.

$\lim_{n\to\infty} \frac{b_n}{a_n} = 0$ if $\sum a_n$ converges, then $\sum b_n$converges.

$\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$ if $\sum a_n$ diverges, then $\sum b_n$ diverges.

$\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$

To find out what series to compare it to, use Bill Gates.

Behaves like $\frac{2n}{n^2} = \frac{2}{n}$ which diverges.

$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{\frac{2n+1}{(n+1)^2}}{\frac{2}{n}} = \frac{(2n+1)n}{2(n+1)^2} = \frac{2n^2+n}{2n^2+4n+2}$ by l'hopital's rule goes to $\frac{4n+1}{4n+4} = 1$

Both series diverge or converge, both of these diverge.

$\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

What does this series approach? $\frac{1}{2^n}$

This is a geometric series with $r = \frac{1}{2}$ which converges. Use this as the comparison series.

$\lim_{n\to\infty} \frac{\frac{1}{2^n}}{\frac{1}{2^n-1}} = \frac{2^n}{2^n-1} = \frac{\frac{2^n}{2^n}}{\frac{2^n}{2^n}-\frac{1}{2^n}} = \frac{1}{1-2^n} = 1$

Both diverge or converge so both converge.

10) The Alternating Series

The series $\sum_{n=1}^{\infty}(-1)^n a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ converges if

all three of the conditions are true:

1) Each $a_n$is positive (the series has to alternate between + and -).
2) $a_n \geq a_{n+1}$ (the terms get smaller left to right)
3) $\lim_{n\to\infty} a_n \to 0$ (The last term has to approach 0.)

$\sum_{n=1}^{\infty}\frac{1}{n}$ is called the harmonic series which diverges.

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is called the alternating harmonic series which converges.

Endpoint Convergence

We need to use the Ratio Test to find the interval of convergence.

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{2n} \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{(n+1)}\frac{x^{2(n+1)}}{2(n+1)}}{(-1)^n\frac{x^{2n}}{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{2n+2}\cdot\frac{2n}{x^{2n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{x^{2n}}\cdot\frac{2n}{2n+2}\right| = \lim_{n\to\infty}|x^2\cdot 1| = x^2 < 1$$

The series converges for $-1 < x < 1$.

Look at the endpoints:

-1	1
$(-1)^{n+1}\frac{(-1)^{2n}}{2n}$	$(-1)^n\frac{1^{2n}}{2n}$
$\frac{(-1)^{3n+1}}{2n}$	$\frac{(-1)^n}{2n}$
This is an alternating harmonic series multiplied by ½.	This is the same series.

Both series converge at the endpoints so the interval of convergence is

[-1, 1].

Does each series converge or diverge? Give the test that justifies your answer.

$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}} > \frac{1}{\sqrt{n}}$ Direct comparison with $\frac{1}{\sqrt{n}}$ which diverges by the p-series.

OR

By the integral test, $\int_1^{\infty} \frac{3}{\sqrt{x}} dx = 6\sqrt{x}\Big|_1^{\infty}$ This diverges so the series diverges.

$\sum_{n=1}^{\infty} \frac{1}{2n-1}$ This approaches $\frac{1}{2n}$. Use the limit comparison test.

$$\lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{2n}} = \lim_{n\to\infty} \frac{1}{2n-1} \cdot \frac{2n}{1} = \lim_{n\to\infty} \frac{2n}{2n-1} = 1$$

$\frac{1}{2n}$ diverges by the integral test so $\frac{1}{2n-1}$ also diverges

$\sum_{n=1}^{\infty} \frac{1}{(ln3)^n}$ This is a geometric series with $r = \frac{1}{ln3} < 1$ so it converges.

$\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

$\sum_{n=1}^{\infty} \frac{5n^3-3n}{n^2(n+2)(n^2+5)} = \lim_{n\to\infty} \frac{5n^3-3n}{n^5+\cdots} = 5 \neq 0$ The nth term test says this diverges.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$ Alternating test says this converges.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$ approaches $(-1)^{n+1} \frac{\sqrt{n}}{n}$ approaches $(-1)^{n+1} \frac{1}{\sqrt{n}}$

Which converges by the alternating series.

$$\lim_{n\to\infty} \frac{(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}}{(-1)^{n+1}\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}+1}{n+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}+1}{n+1} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{n+\sqrt{n}}{n+1} = 1$$

By the limit comparison test, $(-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$ also converges.

$\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n^2})$ By the integral test, $\int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = (lnx + \frac{1}{x})\Big|_1^{\infty}$ diverges.

Homework 7 – 21 odd, 35 – 49 odd (just find the interval of convergence)

7) $\sum_{n=1}^{\infty} \frac{5}{n+1}$ Use Limit Comparison test. $\lim_{n\to\infty} \frac{\frac{5}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{5n}{n+1} = 5$

Both diverge.

Use integral test. $\int_1^{\infty} \frac{5}{x+1} dx = 5lnx + 1| \begin{matrix} \infty \\ 1 \end{matrix}$ which diverges.

9) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ Use integral test. $\int_1^{\infty} \frac{\ln x}{x} = (lnx)^2 \Big| \begin{matrix} \infty \\ 1 \end{matrix}$ which diverges.

11) $\sum_{n=1}^{\infty} \frac{1}{(ln2)^n}$ This is a geometric series with $\frac{1}{ln2} > \frac{1}{lne} = 1$ which diverges.

13) $\sum_{n=1}^{\infty} nsin\left(\frac{1}{n}\right)$ This diverges by the nth term test as

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \frac{-\frac{1}{n^2} cos\frac{1}{n}}{-\frac{1}{n^2}} = \cos\left(\frac{1}{\infty}\right) = cos0 = 1$$

15) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ Use the Limit Comparison test.

$\frac{\sqrt{n}}{n^2+1}$ approaches $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$ which converges by the p-series test.

$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

Both series converge.

17) $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$ Use the nth term test.

$\lim_{n\to\infty} \frac{3^{n-1}+1}{3^n} = \frac{3^{n-1}}{3^n} = \frac{1}{3}$ which is not 0, so it diverges.

19) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{10^n}{n^{10}}$

This looks like an alternating series which converges but do the terms get smaller?

$\frac{10}{1} - \frac{100}{2^{10}} + \frac{1000}{3^{10}} - \cdots$

$10 - .097 + .017 - \cdots$

All terms get smaller, they alternate signs and the last term approaches 0. It converges.

21) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{\ln n^2} \;=\; \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{2\ln n} \;=\; \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\mathrm{n}\, n}{\ln n^2} = \;(-1)^{n+1}(\frac{1}{2})$

This is an alternating series but the nth term does not approach 0. It diverges.

35) $\sum_{n=0}^{\infty} x^n \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n}\right| = |x| < 1$

$-1 < x < 1$

Try 1 $\quad 1^n$ This diverges by the nth term test.

Try -1 $\quad (-1)^n$ This series diverges by the nth term test.

The interval is $-1 < x < 1$.

37) $\sum_{n=0}^{\infty}(-1)^n(4x+1)^n \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(4x+1)^{n+1}}{(-1)^n(4x+1)^n}\right| = \lim_{n\to\infty}|4x+1|$

$-1 < 4x+1 < 1 \qquad -2 < 4x < 0 \qquad -\frac{1}{2} < x < 0$

Try $\frac{1}{2}$ $\quad (-1)^n(4(-\frac{1}{2})+1)^n = (-1)^n(-3^n)$ This diverges by the nth term test.

Try 0 $\quad (-1)^n(4(0)+1)^n = (-1)^n(1)$ This diverges by the nth term test.

The interval is -1/2 < x < 0.

39) $\sum_{n=0}^{\infty}\frac{(x-2)^n}{10^n} \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-2)^{n+1}}{10^{n+1}}}{\frac{(x-2)^n}{10^n}}\right| = \left|\frac{(x-2)^{n+1}}{(x-2)^n}\cdot\frac{10^n}{10^{n+1}}\right| = \left|\frac{x-2}{10}\right| < 1$

$-1 < \frac{x-2}{10} < 1 \qquad -10 < x-2 < 10 \qquad -8 < x < 12$

Try – 8 $\quad \frac{(-8-2)^n}{10^n} = \frac{(-10)^2}{10^n} = (-1)^n\frac{10^n}{10^n} = (-1)^n$ diverges

Try 12 $\quad \frac{(12-2)^n}{10^n} = \frac{10^n}{10^n} = 1$ Diverges

The interval is -8 < x < 12.

41) $\sum_{n=0}^{\infty} \frac{x^n}{n\sqrt{n}\ 3^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)\sqrt{n+1}\ 3^{n+1}}}{\frac{x^n}{n\sqrt{n}\ 3^n}}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)\sqrt{n+1}\ 3^{n+1}} \cdot \frac{3^n n\sqrt{n}}{x^n}\right|$$

$$\lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{n\sqrt{n}}{(n+1)\sqrt{n+1}}\right| = \left|\frac{x}{3}\right| < 1$$

$$-1 < \frac{x}{3} < 1 \qquad -3 < x < 3$$

Try -3 $\quad \frac{(-3)^n}{n\sqrt{n}\ 3^n} = \frac{(-1)^n}{n\sqrt{n}} = \frac{(-1)^n}{n^{\frac{3}{2}}}$ alternating series converges

Try 3 $\quad \frac{3^n}{n\sqrt{n}\ 3^n} = \frac{1}{n\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$ p-series p > 1 converges

The interval is $-3 \le x \le 3$.

43) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)(x+3)^{n+1}}{5^{n+1}}}{\frac{n(x+3)^n}{5^n}}\right| = \lim_{n\to\infty} \left|\frac{(n+1)(x+3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n(x+3)^n}\right|$$

$$\lim_{n\to\infty} \left|\frac{n+1}{n} \cdot \frac{(x+3)^{n+1}}{(x+3)^n} \cdot \frac{5^n}{5^{n+1}}\right| = \left|\frac{x+3}{5}\right| < 1$$

$$-1 < \frac{x+3}{5} < 1 \qquad -5 < x+3 < 5 \qquad -8 < x < 2$$

Try -8 $\quad \frac{n(-8+3)^n}{5^n} = \frac{n(-5)^n}{5^n} = (-1)^n n$ Goes to infinity, diverges.

Try 2 $\quad \frac{n(2+3)^n}{5^n} = \frac{n(5)^n}{5^n} = n$ Goes to infinity, diverges.

The interval is $-8 < x < 2$.

45) $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{\sqrt{n+1}x^{n+1}}{3^{n+1}}}{\frac{\sqrt{n}x^n}{3^n}}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{n+1}x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n}x^n}\right| =$

$\lim_{n\to\infty} \left|\frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{3^n}{3^{n+1}}\right| = \left|\frac{x}{3}\right| < 1$

$-1 < \frac{x}{3} < 1 \quad -3 < x < 3$

Try -3 $\frac{\sqrt{n}(-3)^n}{3^n} = (-1)^n\sqrt{n}$

This does not approach 0. By the nth term test, this diverges.

Try 3 $\frac{\sqrt{n}(3)^n}{3^n} = \sqrt{n}$

This does not approach 0. By the nth term test, this diverges.

The interval is $-3 < x < 3$.

47) $\sum_{n=0}^{\infty}(-2)^n(n+1)(x-1)^n$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-2)^{n+1}(n+2)(x-1)^{n+1}}{(-2)^n(n+1)(x-1)^n}\right|$

$\lim_{n\to\infty} \left|\frac{(-2)^{n+1}}{(-2)^n} \cdot \frac{n+2}{n+1} \cdot \frac{(x-1)^{n+1}}{(x-1)^n}\right| = |2(x-1)| < 1$

$-1 < 2(x-1) < 1 \qquad -\frac{1}{2} < x - 1 < \frac{1}{2} \qquad \frac{1}{2} < x < \frac{3}{2}$

Try ½ $(-2)^n(n+1)(\frac{1}{2} - 1)^n = n + 1$

This does not approach 0. By the nth term test, this diverges.

Try $\frac{3}{2}$ $(-2)^n(n+1)(\frac{3}{2} - 1)^n = (-1)^n(n+1)$

This does not approach 0. By the nth term test, this diverges.

The interval is $\frac{1}{2} < x < \frac{3}{2}$.

49) $\sum_{n=1}^{\infty} \frac{(x+\pi)^n}{\sqrt{n}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+\pi)^{n+1}}{\sqrt{n+1}}}{\frac{(x+\pi)^n}{\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\frac{(x+\pi)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(x+\pi)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x+\pi)^{n+1}}{(x+\pi)^n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}}\right| = |x+\pi| < 1$$

$-1 < x + \pi < 1 \qquad -1 - \pi < x < 1 - \pi$

Try $-1-\pi$ $\quad \frac{(-1-\pi+\pi)^n}{\sqrt{n}} = \frac{(-1)^n}{\sqrt{n}}$ Alternating series converges

Try $1-\pi$ $\quad \frac{(1-\pi+\pi)^n}{\sqrt{n}} = \frac{1}{\sqrt{n}}$ P-series $p < 1$ diverges.

The series is $-1-\pi \le x < 1-\pi$.