New 10.2 Notes

The last section was based on using a geometric series to create an approximation for a curve that would be accurate given an interval of convergence.

Before we get fancy, here is another way to create a power series.

$$y = \frac{6}{1-x}$$

Use long division.

$$\underline{6 + 6x + 6x^2 + \ }$$

$$1 - x \quad)\ 6$$

$$\underline{6 + 6x}$$

$$6x$$

$$\underline{6x + 6x^2}$$

$$6x^2 + 6x^3$$

This creates $y = 6 + 6x + 6x^2 + 6x^3 + \cdots + 6x^n$

Using the Geometric method where $r = x$ and $a_1 = 6$, you get the same thing.

We are now going to create a series based on the derivatives of the function and centered around a given value of x.

What if $y'''(0) = 6, y''(0) = 16, y'(0) = 7 \& \ y(0) = -2$?

$$\int y''' dx = \int 6dx = 6x + C$$

$y''(0) = 6x + C \qquad 16 = 6(0) + C \qquad 16 = C \qquad y''(x) = 6x + 16$

$$\int (6x + 16)dx = 3x^2 + 16x + C$$

$y'(0) = 3x^2 + 16x + C \qquad y'(0) = 7 = \ 3x^2 + 16x + C \quad C = 7$

$y'(0) = \ 3x^2 + 16x + 7$

$$\int (3x^2 + 16x + 7)dx = x^3 + 8x^2 + 7x + C$$

$y(0) = x^3 + 8x^2 + 7x + C = -2$

$y = x^3 + 8x^2 + 7x - 2$

Let's check it to see if it works.

$$y = x^3 + 8x^2 + 7x - 2 \quad y(0) = -2$$

$$y' = 3x^2 + 16x + 7 \qquad y'(0) = 7$$

$$y'' = 6x + 16 \quad y''(0) = 16$$

$$y''' = 6 \qquad y'''(0) = 6$$

Is there a way to just go straight to the function given the derivatives?

$$\frac{-2}{0!} + \frac{7x}{1!} + \frac{16x^2}{2!} + \frac{6x^3}{3!} = -2 + 7x + 8x^2 + x^3$$

This is called a Taylor Polynomial of order three at $x = 0$.

The official formula for the Taylor Polynomial of order n centered at x = 0 is :

$$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!} + \cdots f^{(n)}(0)\frac{x^n}{n!} + \cdots$$

If the Taylor Polynomial is centered at $x = a$, it looks like this:

$$P_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{3!} + f''''(a)\frac{(x-a)^4}{4!} + \cdots f^{(n)}(a)\frac{(x-a)^n}{n!} + \cdots$$

Try $y = e^x$centered at $x = 0$.

$y = e^x \qquad y(0) = 1$

$y' = e^x \qquad y'(0) = 1$

$y'' = e^x \qquad y''(0) = 1$

$y''' = e^x \qquad y'''(0) = 1$

$y'''' = e^x \qquad y''''(0) = 1$

$$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!} + \cdots f^{(n)}(0)\frac{x^n}{n!} + \cdots$$

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \frac{x^n}{n!} + \cdots$$

Try $y = sinx$ centered at $x = 0$.

$y = sinx \qquad y(0) = 0$

$y' = cosx \qquad y'(0) = 1$

$y'' = -sinx \qquad y''(0) = 0$

$$y''' = -cosx \quad y'''(0) = -1$$

$$y'''' = sinx \quad y''''(0) = 0$$

$$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!} + \cdots f^{(n)}(0)\frac{x^n}{n!} + \cdots$$

$$P_n(x) = 0 + x + 0\left(\frac{x^2}{2}\right) + (-1)\frac{x^3}{3!} + (0)\frac{x^4}{4!} + \cdots \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$P_n(x) = x - \frac{x^3}{3!} + \cdots \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Try $y = cosx$ centered at $x = 0$.

$$y = cosx \quad y(0) = 1$$

$$y' = -sinx \quad y'(0) = 0$$

$$y'' = -cosx \quad y''(0) = -1$$

$$y''' = sinx \qquad y'''(0) = 0$$

$$y'''' = cosx \qquad y''''(0) = 1$$

$$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f''''(0)\frac{x^4}{4!} + \cdots f^{(n)}(0)\frac{x^n}{n!} + \cdots$$

$$P_n(x) = 1 + (0)x - \left(\frac{x^2}{2}\right) + (0)\frac{x^3}{3!} + (1)\frac{x^4}{4!} + \cdots \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Could you derive the cosine function by using the sine function?