Free Response Questions for Taylor or Maclaurin Series

1) 1990

Let f be the function defined by $f(x) = \frac{1}{x-1}$.

a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

Use a geometric series. $f(x) = \frac{1}{x-1} = \frac{1}{1+(x-2)}$.

$$f(x) = 1 - (x-2) + (x-2)^2 - (x-2)^3 + \cdots + (-1)^{n+1}(x-2)^n$$

Use Taylors

$f(x) = \frac{1}{x-1} = (x-1)^{-1}$ $\quad$ $f(2) = \frac{1}{2-1} = 1$

$f'(x) = -(x-1)^{-2} = -\frac{1}{(x-1)^2}$ $\quad$ $f'(2) = -1$

$f''(x) = 2(x-1)^{-3} = \frac{2}{(x-1)^3}$ $\quad$ $f''(2) = 2$

$f'''(x) = -6(x-1)^{-4} = \frac{-6}{(x-1)^4}$ $\quad$ $f'''(2) = -6$

$$f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$f(2) + \frac{(-1)}{1!}(x-2) + \frac{(2)f''(2)}{2!}(x-2)^2 + \frac{(-6)f'''(2)}{3!}(x-2)^3$$

$$1 - (x-2) + (x-2)^2 - (x-2)^3 + \cdots + (-1)^{n+1}(x-2)^n$$

b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $ln|x - 1|$.

$$\ln(x-1) = -\int_1^{\infty} f(x)dx$$

$$C + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} - \frac{(x-2)^4}{4} + \cdots + \frac{(-1)^{n+1}(x-2)^{n+1}}{n+1}$$

When $x = 2 \quad \ln(x-1) = \ln(2-1) = ln1 = 0, C = -2$

$$\ln(x-1) = -2 + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} - \frac{(x-2)^4}{4} + \cdots + \frac{(-1)^{n+1}(x-2)^{n+1}}{n+1}$$

2) 1996 .

The Maclaurin series for $f(x)$ is given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$

a) Find $f'(0)$ and $f^{(17)}(0)$.

$$f'(0)\frac{x}{1!} = \frac{x}{2!}, f'(0) = \frac{1}{2}$$

$$f^{(17)}(0)\frac{x^{17}}{17!} = \frac{x^{17}}{18!} \qquad f^{(17)}(0) = \frac{1}{18}$$

b) Let $g(x) = xf(x)$. Write the Maclaurin series for $g(x)$, showing the first three nonzero terms and the general term.

$$g(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!} + \cdots$$

c) Write $g(x)$ in terms of a familiar function without using series. The write $f(x)$ in terms of the same familiar function.

$$g(x) = e^x - 1$$

$$f(x) = \frac{e^x - 1}{x}$$

3) 1979

Let f be the function defined by $f(x) = \frac{1}{1-2x}$

a) Write the first four terms and the general term of the Taylor expansion of $f(x)$ about $x = 0$.

Geometric Series $\quad f(x) = 1 + (2x) + (2x)^2 + (2x)^3 + \cdots + (2x)^n$

Taylor Series

$f(x) = \frac{1}{1-2x} = (1-2x)^{-1} \qquad f(0) = \frac{1}{1-2(0)} = 1$

$f'(x) = 2(1-2x)^{-2} \qquad f'(0) = 2$

$f''(x) = 8(1-2x)^{-3} \qquad f''(0) = 8$

$f'''(x) = 48(1-2x)^{-4} \qquad f'''(0) = 48$

$f(x) = 1 + 2x + \frac{8x^2}{2} + \frac{48x^3}{6} + \cdots +$

$$f(x) = 1 + 2x + 4x^2 + 8x^3 + \cdots + 2^n x^n$$

4) 1991

Let f be the function given by $f(t) = \frac{4}{1+t^2}$ and G be the function given by $(x) = \int_0^x f(t)dt$.

a) Find the first four nonzero terms and the general term for the power series expansion of $f(t)$ about $t = 0$.

Geometric with $a = 4, r = -t^2$

$f(t) = 4 - 4t^2 + 4t^4 - 4t^6 + \cdots + (-1)^n 4t^{2n} + \cdots$

b) Find the first four nonzero terms of $G(x)$ about $x = 0$.

$$G(x) = 4x - \frac{4x^3}{3} + \frac{4x^5}{5} - \frac{4x^7}{7} + \cdots + (-1)^n \frac{4x^{2n+1}}{2n+1} + \cdots$$

5) 1982

a) Write the Taylor series expansion about $x = 0$ for $f(x) = \ln(1+x)$. Include an expression for the general term.

$$f(x) = \ln(1+x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$

$$f''(x) = -(1+x)^{-2} \quad f''(0) = -1$$

$$f'''(x) = 2(1+x)^{-3} \quad f'''(0) = 2$$

$$f(x) = 0 + x - \frac{x^2}{2} + \frac{2x^3}{6} + \cdots$$

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

b) Use the expression found in part (a) to determine the logarithmic function whose Taylor series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2n}$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x^2)^n}{n}$$

$$\frac{1}{2}\ln(1+x^2) = \ln(\sqrt{1+x^2}\,)$$

6) 1983

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, where $a_0 = 1$ and $a_n = \left(\frac{7}{n}\right) a_{n-1}$ for $n \geq 1$.

a) Find the first four terms and the general term of the series.

$$1 + 7x + \frac{49x^2}{2} + \frac{343x^3}{6} + \cdots + \frac{(7x)^n}{n!} + \ldots$$

b) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, find the value of $f'(1)$.

$$f'(x) = 7 + 7^2 + \frac{7^3}{2!} + \frac{7^4}{3!} + \frac{7^5}{4!} + \cdots \frac{7^{n+1}}{n!}$$

$$f'(1) = 7(1 + 7 + \frac{7^2}{2!} + \frac{7^3}{3!} + \frac{7^4}{4!} + \cdots + \frac{7^n}{n!})$$

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$

$$1 + 7 + \frac{7^2}{2!} + \frac{7^3}{3!} + \frac{7^4}{4!} + \cdots + \frac{7^n}{n!} = e^7$$

$$f'(1) = 7e^7$$

7) 1986

a) Find the first four nonzero terms in the Taylor series expansion about $x = 0$ for $f(x) = \sqrt{1+x}$.

$$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}} \qquad f(0) = 1$$

$$f'(x) = \frac{1}{2\sqrt{1+x}} = \frac{1}{2}(1+x)^{-\frac{1}{2}} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \qquad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}} \qquad f'''(0) = \frac{3}{8}$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$

$$1 + \frac{x}{2} - \frac{1x^2}{4(2!)} + \frac{3x^3}{(8)3!} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

b) Use the series result in part (a) to find the first four nonzero terms in the Taylor series expansion about $x = 0$ for $g(x) = \sqrt{1+x^3}$.

$$g(x) = 1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16}$$

c) Find the first four nonzero terms in the Taylor series expansion about $x = 0$ for the function h such that $h'(x) = \sqrt{1+x^3}$ and $h(0) = 4$.

$$h(x) = \int g(x)$$

$$h(x) = 4 + \ x + \frac{x^4}{8} - \frac{x^7}{56}$$

8) 1987

a) Find the first five terms in the Taylor series about $x = 0$ for $f(x) = \frac{1}{1-2x}$.

Geometric

$$r = 2x \qquad f(x) = 1 + (2x) + (2x)^2 + (2x)^3 + (2x)^5$$

b) Use partial fractions and the result from part (a) to find the first five terms in the Taylor series about $x = 0$ for $g(x) = \frac{1}{(1-2x)(1-x)}$.

$$\frac{1}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}$$

$$A(1-x) + B(1-2x) = 1$$

Let $x = 1 \quad -B = 1 \quad B = -1$

Let $x = \frac{1}{2} \quad \frac{1}{2}A = 1 \quad A = 2$

$$\frac{1}{(1-2x)(1-x)} = \frac{2}{1-2x} - \frac{1}{1-x}$$

$$\mathrm{g(x)} = 2(1 + (2\mathrm{x}) + (2\mathrm{x})^2 + (2x)^3 + (2x)^4) - (1 + x + x^2 + x^3 + x^4)$$

$$g(x) = 1 + 3x + 7x^2 + 15x^3 + 31x^4$$

9) 1993

Let f be the function given by $f(x) = e^{\frac{x}{2}}$.

a) Write the first four nonzero terms and the general term for the Taylor series expansion of $f(x)$ about $x = 0$.

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$$e^{\frac{x}{2}} = 1 + \frac{x}{2} + \frac{(\frac{x}{2})^2}{2!} + \frac{(\frac{x}{2})^3}{3!} + \cdots + \frac{(\frac{x}{2})^n}{n!}$$

$$1 + \frac{x}{2} + \frac{x^2}{(4)2!} + \frac{x^3}{(8)3!} + \cdots + \frac{x^n}{2^n n!}$$

b) Use the result from part (a) to write the first three nonzero terms and the general term of the series expansion about $x = 0$ for $g(x) = \frac{e^{\frac{x}{2}}-1}{x}$.

$$e^{\frac{x}{2}} = \qquad 1 + \frac{x}{2} + \frac{x^2}{(4)2!} + \frac{x^3}{(8)3!} + \cdots + \frac{x^n}{2^n n!}$$

$$g(x) = \frac{e^{\frac{x}{2}}-1}{x} = \frac{\left(1+\frac{x}{2}+\frac{x^2}{(4)2!}+\frac{x^3}{(8)3!}+\cdots+\frac{x^n}{2^n n!}\right)-1}{x}$$

$$g(x) = \left(\frac{x}{2x} + \frac{x^2}{(4)2!x} + \frac{x^3}{(8)3!x} + \cdots + \frac{x^n}{2^n n!x}\right)$$

$$g(x) = \left(\frac{1}{2} + \frac{x}{(4)2!} + \frac{x^2}{(8)3!} + \cdots + \frac{x^{n-1}}{2^n n!}\right)$$

c) For the function g in part (b), find $g'(2)$ and use it to show that

$$\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$$

$$g(x) = \frac{e^{\frac{x}{2}}-1}{x}$$

$$g'(x) = \frac{x\left(\frac{1}{2}e^{\frac{x}{2}}\right)-1(e^{\frac{x}{2}}-1)}{x^2}$$

$$g'(2) = \frac{2\left(\frac{1}{2}e^{\frac{2}{2}}\right)-1(e^{\frac{2}{2}}-1)}{2^2} = \frac{e-e+1}{4} = \frac{1}{4}$$

$$g(x) = \left(\frac{1}{2} + \frac{x}{(4)2!} + \frac{x^2}{(8)3!} + \cdots + \frac{x^{n-1}}{2^n n!}\right)$$

$$g'(x) = \left(\frac{1}{(4)2!} + \frac{2x}{(8)3!} + \cdots + \frac{(n-1)x^{n-2}}{2^n n!}\right)$$

$$g'(2) = \left(\frac{1}{(4)2!} + \frac{2(2)}{(8)3!} + \cdots + \frac{(n-1)2^{n-2}}{2^n n!}\right)$$

$$g'(2) = \frac{1}{8} + \frac{1}{12} + \cdots + \frac{n-1}{4n!} = \sum_{n=1}^{\infty} \frac{n}{4(n+1)!}$$

SO………

$$g'(2) = \sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$$

10) 1994

Let f be the function given by $f(x) = e^{-2x^2}$.

a) Find the first four nonzero terms and the general term of the power series for $f(x)$ about $x = 0$.

Since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$$e^{-2x^2} = 1 + (-2x^2) + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \cdots + \frac{(-2x^2)^n}{n!}$$

$$e^{-2x^2} = 1 - 2x^2 + 2x^4 - \frac{4x^6}{3} + \cdots + \frac{(-2x^2)^n}{n!}$$

11) 2002

The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

$$f'(x) = 2 + 4x + 8x^2 + 16x^3 + \cdots + 2(2x)^n + \cdots$$

c) Use the Macluarin series you found in part (b) to find the value of $f'(-\frac{1}{3})$.

Geometric Method

$$f'(x) = \frac{2}{1-2x} \qquad f'\left(-\frac{1}{3}\right) = \frac{2}{(1-2(-\frac{1}{3})} = \frac{2}{1+\frac{2}{3}} = \frac{6}{5}$$

Taylor Series

$$f'(x) = 2 + 4(-\frac{1}{3}) + 8(-\frac{1}{3})^2 + 16(-\frac{1}{3})^3 + \cdots + 2(2(-\frac{1}{3}))^n + \cdots$$

$$= 2 - \tfrac{4}{3} + \tfrac{8}{9} - \tfrac{16}{27} + \cdots + 2(-\tfrac{2}{3})^n + \cdots$$

$$= \frac{2}{1+\frac{2}{3}} = \frac{6}{5}$$

12) 2003

The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

$f'(0)$ is the coefficient of the x term $= 0$

$f''(0) = 2$(the coefficient of the x^2 term) $= 2\left(-\frac{1}{3!}\right) = -\frac{1}{3}$

f has a local maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$

b) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = cosx$.

$$y = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots + \frac{(2n)(-1)^n x^{2n-1}}{(2n+1)!}$$

$$xy' = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots + \frac{(2n)(-1)^n x^{2n}}{(2n+1)!}$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{2n!}$$

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots+\frac{(-1)^n x^{2n}}{(2n+1)!}+-\frac{2x^2}{3!}+\frac{4x^4}{5!}-\frac{6x^6}{7!}+\cdots+\frac{(2n)(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^n x^{2n}}{2n!}$$

13) 2005

Let f be a function with derivatives of all orders and for which $f(2)=7$. When n is odd, the nth derivative of f at $x=2$ is given by $f^{(n)}(2)=\frac{(n-1)!}{3^n}$.

a) Write the sixth-degree Taylor polynomial for f about $x=2$.

b) In the Taylor series for f about $x=2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?