Error

 Actual - approximation

 Geometric series

 Find the error using just the first three terms of

 $\sum\_{n=1}^{\infty }\frac{5}{7}\left(\frac{2}{3}\right)^{n}$

 First three terms = $\frac{10}{21}+\frac{20}{63}+\frac{40}{189}=1.005291$

 Actual = $\frac{\frac{10}{21}}{1 - \frac{2}{3}}=\frac{\frac{10}{21}}{\frac{1}{3}}=\frac{30}{21}=\frac{10}{7}=1.428571$

 Error = .423280

Alternating series

 Find the error of using just the first three terms of

 $\sum\_{n=1}^{\infty }\left(\frac{-1}{n}\right)^{n}$

 $-\frac{1}{1}+\frac{1}{4}-\frac{1}{27}=-.787037$

 The error is less than the next term = $\frac{1}{4}$

 Find the error of using just the first four terms of

 $\sum\_{n=1}^{\infty }\frac{cos⁡(nπ)}{n}$

 $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}=-.583333$

 The error is less than the next term = $\frac{1}{5}=.2$ .

Maclaurin’s Theorem

 $f\left(0\right)+f^{'}\left(0\right)x+\frac{f^{''}(0)x^{2}}{2!}+\frac{f^{'''}(0)x^{3}}{3!}+\frac{f^{\left(4\right)}(0)x^{4}}{4!}+…+\frac{f^{\left(n\right)}(0)x^{n}}{n!}+…$

Maclaurin Series with Lagrange Remainder

$f\left(0\right)+f^{'}\left(0\right)x+\frac{f^{''}(0)x^{2}}{2!}+\frac{f^{'''}(0)x^{3}}{3!}+\frac{f^{\left(4\right)}(0)x^{4}}{4!}+…+\frac{f^{\left(n\right)}(0)x^{n}}{n!}+\frac{f^{\left(n+1\right)}(c)x^{n+1}}{(n+1)!}$

Lagrange Error Bound

If $R\_{n}\left(x\right)=\frac{f^{\left(n+1\right)}(c)x^{n+1}}{(n+1)!}$ is the Lagrange remainder after partial sum $S\_{n}(x)$, and $M$ is an upper bound for $\left|f^{\left(n+1\right)}(x)\right|$ on the interval between $0 and x$, then

A Lagrange error bound for $S\_{n}(x)$ is given by

$$\left|R\_{n}(x)\right|\leq \left|\frac{M}{\left(n+1\right)!}x^{n+1}\right|$$

$M$ is the largest possible value for the $ (n+1$) st derivative between $0 and x.$

There can be many answers for the error bound.

Example 1 when I give you the $M$

Suppose that the first five terms of the Maclaurin series for $f(0.2)$ for a function $f$ are $S\_{4}\left(0.2\right)=9+4\left(0.2\right)+2\left(0.2^{2}\right)-9\left(0.2^{3}\right)+.098(0.2^{4})$

1. Find the value of the fourth derivative, $f^{\left(4\right)}(0.2)$.

$$\frac{f^{\left(4\right)}\left(.2\right)(-.2)^{4}}{4!}=.098(0.2^{4})$$

$f^{\left(4\right)}\left(0.2\right)=.098\left(4!\right)=$2.352

1. Given that $\left|f^{5}(0.2)\right|\leq 0.09$, show that the remainder of the series after the first five terms has an absolute value less than $10^{-5}.$

$$R\_{4}\leq \left|\frac{M(.2)^{5}}{5!}\right|\leq \left|\frac{.09(.2)^{5}}{5!}\right|=2.4 ∙10^{-7}<10^{-5}$$

Example \* when I give you the $M$

Suppose that the first six terms of the Maclaurin series for $f(0.6)$ for a function $f$ are $S\_{5}\left(0.2\right)=19-6\left(0.6\right)+\left(0.6^{2}\right)-9\left(0.6^{3}\right)+3.6\left(0.6^{4}\right)-7.4(0.6^{5})$

1. Find the value of the fifth derivative, $f^{\left(5\right)}(0.6)$.

$$\frac{f^{5}\left(0.6\right)(.6)^{5}}{5!}=-7.4(0.6^{5})$$

$$ f^{\left(5\right)}\left(0.6\right)=5!\left(-7.4\right)=-888$$

1. Given that $\left|f^{6}(0.6)\right|\leq 0.04$, show that the remainder of the series after the first five terms has an absolute value less than $10^{-5}.$

$$R\_{4}\leq \left|\frac{M(.4)^{6}}{6!}\right|\leq \left|\frac{.04(.4)^{6}}{6!}\right|=4.551111$$

Example 1 when you have to figure out the M

1. Find the first four non-zero terms of the Maclaurin series for the function

$f\left(x\right)=sin⁡(x)$.

 $sinx=x-\frac{x^{3}}{6}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}$

1. Find $sin⁡(.2)$ using this approximation using 6 places past the decimal.

$$\sin(\left(.2\right))=.2-\frac{\left(.2^{3}\right)}{6}+\frac{\left(.2^{5}\right)}{5!}-\frac{\left(.2^{7}\right)}{7!}$$

1. What is the actual value of $sin⁡(.2)$?
2. What is the actual error using problem b as your approximation.
3. Find the error bound of problem b using the fact that this is an alternating series.
4. Find the Lagrange error bound of this approximation.

Example 2 when you have to figure out the M

1. Find the first four non-zero terms of the Maclaurin series for the function

$$f\left(x\right)=e^{-x}$$

 $e^{-x}=1-x+\frac{x^{2}}{2}-\frac{x^{3}}{6} $

1. Find $e^{-.3}$ using this approximation using 6 places past the decimal.

$$e^{-.3}=1-\left(.3\right)+\frac{\left(.3^{2}\right)}{2}-\frac{\left(.3^{3}\right)}{6}$$

1. What is the actual value of $e^{-.3}$?
2. What is the actual error using problem b as your approximation.
3. Find the error bound of problem b using the fact that this is an alternating series.
4. Find the Lagrange error bound of this approximation.

More Examples

1. Find the fourth degree Taylor polynomial for $cosx$ about $x=0.$

$$cosx=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}$$

* 1. Use your polynomial to approximate the value of $\cos(\left(.8\right)).$ Use 6 places.

$$cos⁡(.8)=1-\frac{(.8)^{2}}{2!}+\frac{(.8)^{4}}{4!}$$

* 1. Use the Lagrange Error bound to determine the accuracy of the approximation.

$$Error \leq \left|\frac{M}{\left(n+1\right)!}(x)^{n+1}\right|$$

 $$Error \leq \left|\frac{1}{\left(4+1\right)!}(.8)^{4+1}\right|$$

1. Write the second-degree polynomial for $f\left(x\right)=x^{\frac{5}{2}}$ centered at $x=1$

$$f\left(x\right)=x^{\frac{5}{2}} f\left(1\right)=\frac{5}{2}$$

 $f^{'}\left(x\right)=\frac{5}{2}x^{\frac{3}{2}} f^{'}\left(1\right)=\frac{3}{2}$

 $f^{''}\left(x\right)=\frac{15}{4}x^{\frac{1}{2}} f^{''}\left(1\right)=\frac{15}{4}$

$$f\left(x\right)=\frac{5}{2}+\frac{3}{2}\left(x-1\right)+\frac{15}{4}(x-1)^{2}$$

 2.4) Use your polynomial to approximate $f\left(1.7\right).$ Use 6 places.

$$f\left(1.7\right)=\frac{5}{2}+\frac{3}{2}\left(1.7-1\right)+\frac{15}{4}(1.7-1)^{2}$$

 $$f\left(1.7\right)=\frac{5}{2}+\frac{3}{2}\left(.7\right)+\frac{15}{4}(.7)^{2}$$

 2.8) Find a Lagrange error bound for the maximum error on the approximation.

$$Error \leq \left|\frac{M}{\left(n+1\right)!}(x-1)^{n+1}\right|$$

 $M\leq f^{n+1}(c)$

$f^{'''}\left(x\right)=\frac{15}{8\sqrt{c}}$ where $1\leq c\leq 1.7$ and you want to create the largest possible value

 $c=1$ creates the largest possible value

 $Error\leq \frac{\frac{15}{8}(.7)^{3}}{3!}=$

Let $f$ be a function that has derivatives of all orders for all real numbers $x.$ Assume that $f\left(5\right)=6, f^{'}\left(5\right)=8, f^{''}\left(5\right)=30, f^{'''}\left(5\right)=48$ and

$\left|f^{\left(4\right)}(x)\right|\leq 75$ for all $x$ in the closed interval [5, 5.2].

3) Find the third-degree Taylor polynomial about $x=5$ for $f\left(x\right).$

 $6+8\left(x-5\right)+\frac{30(x-5)^{2}}{2}+\frac{48(x-5)^{3}}{3!}=$

 $6+8\left(x-5\right)+15(x-5)^{2}+6(x-5)^{3}$

3.6) Use this approximation to estimate $f\left(5.2\right).$

 $6+8\left(5.2-5\right)+15(5.2-5)^{2}+6(5.2-5)^{3}$

 $6+8\left(.2\right)+15(.2)^{2}+6(.2)^{3}=6+1.6+.6+.048=8.248$

3.97) What is the maximum possible error in making this estimate?

$$Error \leq \left|\frac{M}{\left(n+1\right)!}(x-5)^{n+1}\right|=\frac{75(.2)^{4}}{4!}=$$

4) Let $f$ be a function that has derivatives of all orders. Assume that

 $f\left(3\right)=1, f^{'}\left(3\right)=\frac{1}{2} , f^{''}\left(3\right)=-\frac{1}{4} , f^{'''}\left(3\right)=\frac{3}{8} $and the graph of $f^{4}(x)$

 on [3, 3.7] is shown below.

 (3.4, 6)

 (3, 4)

 (3.7, 2)

1. Find the Taylor polynomial about $x=3$ for the function $f.$

$$1+\frac{1}{2}\left(x-3\right)-\frac{1}{4}\left(\frac{1}{2}\right)\left(x-3\right)^{2}+\frac{3}{8}\left(\frac{1}{6}\right)\left(x-3\right)^{3}$$

1. Use your answer to approximate the value of $f(3.7)$.

$$f\left(3.7\right)=1+\frac{1}{2}\left(3.7-3\right)-\frac{1}{8}\left(3.7-3\right)^{2}+\frac{1}{16}\left(3.7-3\right)^{3}$$

1. What is the maximum possible error for this approximation ?

$$Error \leq \left|\frac{6}{4!}(3.7-3)^{4}\right|$$

5) Let $f$ be the function defined by $f(x)=\sqrt{x}$.

1. Find the second-degree Taylor polynomial about $x=4$ for the function $f$.

$f\left(x\right)=\sqrt{x}=x^{\frac{1}{2}} f\left(4\right)=2$

 $f^{'}\left(x\right)=\frac{1}{2x^{\frac{1}{2}}}=\frac{1}{2}x^{-\frac{1}{2}} f^{'}\left(4\right)=\frac{1}{8}$

 $f^{''}\left(x\right)=-\frac{1}{4}x^{-\frac{3}{2}} f^{''}\left(4\right)=-\frac{1}{4}\left(\frac{1}{8}\right)=-\frac{1}{32}$

 $f\left(x-4\right)=2+\frac{1}{8}\left(x-4\right)-\frac{1}{32}(\frac{1}{2})\left(x-4\right)^{2}$

 $f\left(x-4\right)=2+\frac{1}{8}\left(x-4\right)-\frac{1}{64}\left(x-4\right)^{2}$

1. Use your answer to approximate the value of $f(5.1)$.

$$f\left(5.1\right)=2+\frac{1}{8}\left(5.1-4\right)-\frac{1}{64}\left(5.1-4\right)^{2}$$

 $f\left(5.1\right)=2+\frac{1}{8}\left(1.1\right)-\frac{1}{64}\left(1.1\right)^{2}$

1. Find a bound on the error for the approximation.

$$Error \leq \left|\frac{M}{\left(n+1\right)!}(x-4)^{n+1}\right|$$

$$M\leq f^{n+1}(c)$$

 $f^{''}\left(x\right)=-\frac{1}{4}x^{-\frac{3}{2}} f^{'''}\left(c\right)=\frac{3}{8}x^{-\frac{5}{2}} where 4\leq c\leq 5.1$

 $c=4$ creates the largest M

 $f^{'''}\left(c\right)=\frac{3}{8}x^{-\frac{5}{2}}=\frac{3}{8}\left(\frac{1}{x}\right)^{\frac{5}{2}}=\frac{3}{8}\left(\frac{1}{4}\right)^{\frac{5}{2}}=\frac{3}{8}\left(\frac{1}{32}\right)=\frac{3}{256}$

 $Error\leq \left|\frac{\frac{3}{256}\left(1.1\right)^{3}}{3!}\right|= $

1. Find the value of $\left|f\left(5.1\right)-P\_{2}(5.1)\right|$

6) Let $f$ be the function given by $f\left(x\right)=cos⁡(3x+\frac{π}{6})$ and let $P(x)$ be the fourth-

 degree Taylor polynomial for $f$ about $x=0.$

1. Find $P(x)$.
2. Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right)-P\left(\frac{1}{6}\right)\right|<\frac{1}{3000}$.