CD 12

1) 2002

The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

a) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

$$f'(x) = 2 + 4x + 8x^2 + 16x^3 + \cdots + (2x)^n$$

b) Use the Maclaurin series you found in part (b) to find the value of $f'(-\frac{1}{3})$.

Go Geo. $a_1 = 2$ $\quad r = 2x \qquad f'\left(-\frac{1}{3}\right) = \frac{2}{1-2(-\frac{1}{3})} = \frac{2}{1+\frac{2}{3}} = \frac{2}{\frac{5}{3}} = \frac{6}{5}$

2) 2003

The function f is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

$$f'(0)x = 0x \quad f'(0) = 0 \qquad \frac{f''(0)x^2}{2!} = -\frac{x^2}{3!} \qquad f''(0) = -\frac{2!}{3!} = -\frac{1}{3}$$

If $f'(0) = 0$ and $f''(0) < 0$, it is concave down ,soc there is a relative maximum at $x = 0$.

b) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = cosx$.

$$y = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

$$y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots + \frac{(-1)^n (2n) x^{2n-1}}{(2n+1)!} + \cdots$$

$$xy' = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots + \frac{(-1)^n (2n) x^{2n}}{(2n+1)!} + \cdots$$

$$y = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

$$xy' + y = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} = cosx$$

3) 2005

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.

a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

$$7 + \frac{x^2}{3^2} + \frac{3!\,x^4}{3^4 4!} + \frac{5!\,x^6}{3^6 6!}$$

$$7 + \frac{x^2}{3^2(2)} + \frac{x^4}{3^4(4)} + \frac{x^6}{3^6(6)}$$

b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

$$\frac{(2n-1)!}{3^{2n}(n!)}$$

4) 2006

The function f is defined by the power series $f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n n x^n}{n+1} + \cdots$

for all real numbers x for which the series converges.

The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n!)}$$

a) The graph of $y = f(x) - g(x)$ passes through $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimuim, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

$$f(x) - g(x) = -1 - \frac{x}{2} + \frac{x}{2} + \frac{2x^2}{3} - \frac{x^2}{4!} + \cdots$$

$$= -1 + \frac{16x^2}{24} - \frac{x^2}{24} + \cdots = -1 + \frac{15x^2}{24} = -1 + \frac{5x^2}{8}$$

Since $f'(0) = 0 \; and \; f''(0) > 0$, there is a local minimum as the function is concave up.

2007

Let f be the function given by $f(x) = e^{-x^2}$.

a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-1)^{n+1}(x)^{2n}}{n!}$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots + \frac{(-1)^{n+1}(x)^{2n}}{n!}$$

b) Use your answer in (a) to find $lim_{x\to 0} \frac{1-x^2-f(x)}{x^4}$.

$$\frac{1 - x^2 - (1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots + \frac{(-1)^{n+1}(x^{2n})}{n!}}{x^4}$$

$$lim_{x\to 0} \frac{1}{2} - \frac{x^2}{3!} + \cdots + \frac{(-1)^{n+1}(x)^{2n-4}}{n!} = \frac{1}{2}$$

c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{\frac{1}{2}} e^{-t^2}dt$.

$$\int_0^x e^{-t^2}dt = x - \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42}$$

$$\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} = \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$$

5) 2008

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having the derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these derivatives are increasing on the interval $1 \leq x \leq 3$.

a) Write the first degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain.

$$h(x) = 80 + 128(x - 2)$$

$$h(1.9) = 80 + 128(1.9 - 2) = 80 + 128(-.1) = 80 - 12.8 = 67.2$$

Since $h''(0) > 0$, h is concave up, so the approximation is lower than the actual value.

b) Write the third degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

$$h(1.9) = 80 + 128(1.9 - 2) + \frac{488(1.9 - 2)^2}{3(2!)} + \frac{448(1.9 - 2)^3}{3(3!)} = 67.988$$

6) 2009

The Maclaurin series for $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$e^{(x-1)^2} = 1 + (x-1)^2 + \frac{((x-1)^2)^2}{2} + \frac{((x-1)^2)^3}{3!} + \cdots + \frac{((x-1)^2)^n}{n!}$$

$$e^{(x-1)^2} = 1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{3!} + \cdots + \frac{((x-1)^2)^n}{n!}$$

b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$$f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2} = \frac{1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{3!} + \cdots + \frac{((x-1)^2)^n}{n!} - 1}{(x-1)^2}$$

$$f(x) = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{3!} + \cdots + \frac{((x-1)^{2n-2}}{n!}$$

c) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

Since $f''(x)$ is always a positive number for all values of x, the concavity is always positive, so it never changes and there are no points of inflection.

7) 2010

$$f(x) = \begin{cases} \frac{cosx - 1}{x^2} & for\ x \neq 0 \\ -\frac{1}{2} & for\ x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $(x) = 1 + \int_0^x f(t)dt$.

a) Write the first three nonzero terms and the general term of the Taylor series for $cosx$ about $x = 0$. Use this series to write the first three nonzero terms of the Taylor series for f about $x = 0$.

$$cosx = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + \frac{(-1)^{n+1}x^{2n}}{2n!}$$

$$cosx - 1 = -\frac{x^2}{2} + \frac{x^4}{4!} + \cdots + \frac{(-1)^{n+1}x^{2n}}{2n!}$$

$$\frac{cosx-1}{x^2} = \frac{-\frac{x^2}{2}+\frac{x^4}{4!}+\cdots+\frac{(-1)^{n+1}x^{2n}}{2n!}}{x^2} = -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!}$$

b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

Since $f'(0) = 0$ & $f''(0) = \frac{1}{12}$, f is concave up so it is a local minimum.

c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

8) 2011

Let $f(x) = \sin(x^2) + cosx$.

a) Write the first four nonzero terms of the Taylor series for $sinx$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin(x^2)$ about $x = 0$.

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}$$

$$sinx^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots + \frac{(-1)^{n+1}x^{4n+2}}{(2n+1)!}$$

$$x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots + \frac{(-1)^{n+1}x^{4n+2}}{(2n+1)!}$$

b) Write the first four nonzero terms of the Taylor series for $cosx$ about $x = 0$. Use this series and the series for $\sin(x^2)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^{n+1}x^{2n}}{(2n)!}$$

$$sinx^2 + cosx = 1 + x^2 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{3!} - \frac{x^6}{6!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!}$$

c) Find the value of $f^{(6)}(0)$.

$$\frac{f^{(6)}(0)x^6}{6!} = -\frac{121x^6}{6!} \qquad f^{(6)}(0) = -121$$

9) 2012

The function g has derivatives of all orders, and the Maclaurin series for g is

$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} + \cdots$

a) Write the first three nonzero terms and the general terms of the Maclaurin series for $g'(x)$.

$$g'(x) = \frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7} + \cdots + \frac{(-1)^n(2n+1)x^{2n}}{(2n+3)}$$