10.4 Converging and Diverging Series

A series converges with three possibilities:

1) It converges within a boundary
2) It converges for any value
3) It converges at only one value

Any sequence will converge at the point where you centered the series.

The best way to find where a series converges is to use the absolute value of the Ratio Test.

THE RATIO TEST

Let $\sum a_n$ be a series with positive terms,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

The series converges if $|L| < 1$.

The series diverges if $L > 1$.

The test does not work if $L = 1$.

We will solve for L to find where our series converges.

Does $\sum_{n=0}^{\infty}\left|\frac{3^n}{5^n+1}\right|$ converge?

$$\lim_{n\to\infty}\frac{\frac{3^{n+1}}{5^{n+1}+1}}{\frac{3^n}{5^n+1}} = \lim_{n\to\infty}\frac{3^{n+1}}{5^{n+1}+1}\cdot\frac{5^n+1}{3^n} = \lim_{n\to\infty}\frac{3^{n+1}}{3^n}\frac{5^n+1}{5^{n+1}+1} = \lim_{n\to\infty}3\left(\frac{1}{5}\right) = \frac{3}{5}$$

Since this value is less than 1, the series converges.

Where does $\sum_{n=0}^{\infty} \left|\frac{nx^n}{10^n}\right|$ converge?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)x^{n+1}}{10^{n+1}}}{\frac{nx^n}{10^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{10^{n+1}}\cdot\frac{10^n}{nx^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{10^n}{10^{n+1}}\cdot\frac{x^{n+1}}{x^n}\right| = \left|\frac{x}{10}\right| =$$

$$\left|\frac{x}{10}\right| < 1 \quad -10 < x < 10$$

Since the series is centered at $x = 0$, the convergence goes from -10 to 10 so we say the radius of convergence is 10 and the interval of convergence is

$-10 < x < 10.$

Find the radius of convergence of the series $\sum_{n=0}^{\infty}|n!\,x^n|$.

$$\lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}|(n+1)x| < 1$$

The only way that $|(n+1)x| < 1$ is if $x = 0$.

Since the series is centered at $x = 0$, the convergence goes from 0 to 0 so we say the radius of convergence is 0 and the interval of convergence is 0.

Find the radius of convergence of the series $\sum_{n=0}^{\infty}\left|\frac{(-x)^n}{n!}\right|$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-x)^{n+1}}{(n+1)!}}{\frac{(-x)^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(-x)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-x)^n}\right| = \lim_{n\to\infty}\left|\frac{(-x)^{n+1}}{(-x)^n}\cdot\frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-x}{n+1}\right| = 0 < 1$$

The series converges for all values of x, so the radius of convergence is ∞.

Converging and Diverging Tests

1) An arithmetic series will always diverge.

2) A geometric series will converge if the $|r| < 1$.

3) The Ratio Test
   Let $\sum a_n$ be a series with positive terms,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

   The series converges if $|L| < 1$.
   The series diverges if $L > 1$.
   The test does not work if $L = 1$.

4) The nth Term Test for Divergence

   $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or does not equal 0.

   Please note that this does not tell you a series converges if the limit = 0.

5) Direct Comparison Test
   Let $\sum a_n$ be a series with no negative terms.

   $\sum a_n$ converges if $\sum b_n$ converges and $a_n < b_n$
   $\sum a_n$ diverges if $\sum c_n$ diverges and $a_n > c_n$

More to come later!!!!!!!

Telescoping series

Find the sum of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Use partial fractions to rewrite the nth term.

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \qquad A(n+1) + Bn = 1$$

Let $n = -1, -B = 1 \quad B = -1$

Let $n = 0, A = 1$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Create a sum using several values of n.

$n = 1 \quad \frac{1}{1} - \frac{1}{2}$

$n = 2 \quad \frac{1}{2} - \frac{1}{3}$

$n = 3 \quad \frac{1}{3} - \frac{1}{4}$

$n = 4 \quad \frac{1}{4} - \frac{1}{5}$

The sum of these terms equal $1 - \frac{1}{n}$. As $n \to \infty$, this goes to 1.

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Fun series to remember. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$

$e = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$

3) $\sum_{n=0}^{\infty} \frac{x^{3n}}{2n!+1}$ $\frac{x^{3n}}{2n!+1} < \frac{x^{3n}}{n!}$

This denominator is larger than this one.

$\frac{x^{3n}}{n!} = \frac{(x^3)^n}{n!} = e^{x^3}$ which converges for all x.

$\frac{x^{3n}}{2n!+1}$ must all converge for all x.

5) $\sum_{n=0}^{\infty} \frac{(cosx)^n}{n!+1}$ $\frac{(cosx)^n}{n!+1} < \frac{(cosx)^n}{n!}$

This denominator is larger than this one.

$(cosx)^n < 1$ so $\frac{(cosx)^n}{n!} < \frac{1}{n!} = e$ which is a value.

So $\sum_{n=0}^{\infty} \frac{(cosx)^n}{n!+1}$ converges.

7) $\sum_{n=0}^{\infty} x^n$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty} |x| < 1$

$-1 < x < 1$ The radius of convergence is 1.

9) $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(4x+1)^{n+1}}{(4x+1)^n}\right| = |4x+1| < 1$

$-1 < 4x + 1 < 1$ $\quad -2 < 4x < 0$ $\quad -\frac{1}{2} < x < 0$

The radius of convergence is $\frac{1}{4}$.

11) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$ $\quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{\frac{(x-2)^{n+1}}{10^{n+1}}}{\frac{(x-2)^n}{10^n}}\right| = \lim\limits_{n\to\infty} \left|\frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n}\right| =$

$\lim\limits_{n\to\infty} \left|\frac{(x-2)^{n+1}}{(x-2)^n} \cdot \frac{10^n}{10^{n+1}}\right| = \left|\frac{x-2}{10}\right| < 1$

$-1 < \frac{x-2}{10} < 1 \quad -10 < x - 2 < 10 \quad -8 < x < 12$

The radius of convergence is 10.

13) $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}3^n}$ $\quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)\sqrt{n+1}3^{n+1}}}{\frac{x^n}{n\sqrt{n}3^n}}\right| = \lim\limits_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)\sqrt{n+1}\,3^{n+1}} \cdot \frac{n\sqrt{n}3^n}{x^n}\right|$

$\lim\limits_{n\to\infty} \left|\frac{x^{n+1}}{x^n} \cdot \frac{n}{n+1} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{3^n}{3^{n+1}}\right| =$

$\lim\limits_{n\to\infty} \frac{x^{n+1}}{x^n} = x \quad \lim\limits_{n\to\infty} \frac{n}{n+1} = 1 \quad \lim\limits_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1 \quad \lim\limits_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{1}{3}$

$\left|\frac{x}{3}\right| < 1 \quad -1 < \frac{x}{3} < 1 \quad -3 < x < 3$

The radius of convergence is 3.

15) $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$ $\quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{\frac{(n+1)(x+3)^{n+1}}{5^{n+1}}}{\frac{n(x+3)^n}{5^n}}\right| =$

$$\lim_{n\to\infty} \left|\frac{(n+1)(x+3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n(x+3)^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)}{n} \cdot \frac{5^n}{5^{n+1}} \cdot \frac{(x+3)^{n+1}}{(x+3)^n}\right| =$$

$\lim\limits_{n\to\infty} \frac{n+1}{n} = 1 \quad \lim\limits_{n\to\infty} \frac{5^n}{5^{n+1}} = \frac{1}{5} \quad \lim\limits_{n\to\infty} \frac{(x+3)^{n+1}}{(x+3)^n} = (x+3)$

$$\left|\frac{x+3}{5}\right| < 1 \quad -1 < \frac{x+3}{5} < 1 \quad -5 < x + 3 < 5 \quad -8 < x < 2$$

The radius of convergence is 5.

17) $\sum_{n=0}^{\infty} n!\,(x-4)^n$ $\quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\frac{(n+1)!(x-4)^{n+1}}{n!(x-4)^n}\right|$

$\lim_{n\to\infty} \frac{(n+1)!}{n!} = n + 1 \quad \lim\limits_{n\to\infty} \frac{(x-4)^{n+1}}{(x-4)^n} = (x-4)$

$\lim_{n\to\infty}(n+1)(x-4) = \infty$ unless x = 4.

The radius of convergence is 0.

19) $\sum_{n=0}^{\infty}(-2)^n(n+1)(x-1)^n$ $\quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(n+2)(x-1)^{n+1}}{(-2)^n(n+1)(x-1)^n}\right|$

$\lim_{n\to\infty}\frac{(-2)^{n+1}}{(-2)^n} = 2 \quad \lim_{n\to\infty}\frac{n+2}{n+1} = 1 \quad \lim_{n\to\infty}\frac{(x-1)^{n+1}}{(x-1)^n} = (x-1)$

$|2(x-1)| < 1 \quad -1 < 2(x-1) < 1 \quad -\frac{1}{2} < x-1 < \frac{1}{2} \quad \frac{1}{2} < x < 1\frac{1}{2}$

The radius of convergence is ½ .

21) $\sum_{n=0}^{\infty}\frac{(x+\pi)^n}{\sqrt{n}}$ $\quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x+\pi)^{n+1}}{\sqrt{n+1}}}{\frac{(x+\pi)^n}{\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\frac{(x+\pi)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(x+\pi)^n}\right|$

$\lim_{n\to\infty}\frac{(x+\pi)^{n+1}}{(x+\pi)^n} = (x+\pi) \quad \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = 1$

$|x+\pi| < 1 \quad -1 < x+\pi < 1 \quad -1-\pi < x < 1-\pi$

The radius of convergence is 1.

23) $\sum_{n=0}^{\infty}\frac{(x-1)^{2n}}{4^n} = \sum_{n=0}^{\infty}\frac{((x-1)^2)^n}{4^n} = \sum_{n=0}^{\infty}\left(\frac{(x-1)^2}{4}\right)^n$

$\left|\frac{(x-1)^2}{4}\right| < 1 \quad -1 < \frac{(x-1)^2}{4} < 1 \quad -4 < (x-1)^2 < 4$

$0 < (x-1)^2 < 4 \quad -2 < x-1 < 2 \quad -1 < x < 3$

The interval of convergence is $-1 < x < 3$.

The sum of the series is $\frac{a}{1-r} = \frac{1}{1-\frac{(x-1)^2}{4}} = \frac{1}{\frac{4-(x-1)^2}{4}} = \frac{4}{4-(x-1)^2}$

25) $\sum_{n=0}^{\infty}\left(\frac{\sqrt{x}}{2}-1\right)^n$ $\quad \left|\frac{\sqrt{x}}{2}-1\right| < 1 \quad -1 < \frac{\sqrt{x}}{2}-1 < 1 \quad 0 < \frac{\sqrt{x}}{2} < 2$

$0 < \sqrt{x} < 4 \quad 0 < x < 16$

The interval of convergence is $0 < x < 16$.

The sum of the series is $\frac{1}{1-\left(\frac{\sqrt{x}}{2}-1\right)} = \frac{1}{2-\frac{\sqrt{x}}{2}} = \frac{1}{\frac{4-\sqrt{x}}{2}} = \frac{2}{4-\sqrt{x}}$.

27) $\sum_{n=0}^{\infty}\left(\frac{x^2-1}{3}\right)^n$ $\left|\frac{x^2-1}{3}\right|<1$ $-1<\frac{x^2-1}{3}<1$ $-3<x^2-1<3$

$-1<x^2<4$ $0<x^2<4$ $-2<x<2$

The interval of convergence is $-2<x<2$.

The sum of the series is $\frac{1}{1-\frac{x^2-1}{3}}=\frac{1}{\frac{3-(x^2-1)}{3}}=\frac{3}{4-x^2}$

29) $\sum_{n=1}^{\infty}\frac{n}{n+1}$ The nth term test says that the series diverges if the nth term does not equal 0. The nth term equals 1.

31) $\sum_{n=1}^{\infty}\frac{n^2-1}{2^n}$ Use the Ratio Test. $\lim_{n\to\infty}(\frac{(n+1)^2-1}{2^{n+1}}\cdot\frac{2^n}{n^2-1})=\lim_{n\to\infty}(\frac{(n+1)^2-1}{n^2-1}\cdot\frac{2^n}{2^{n+1}})$

$\lim_{n\to\infty}\left(\frac{(n+1)^2-1}{n^2-1}\right)=1$ $\lim_{n\to\infty}=\frac{1}{2}$ $\frac{1}{2}<1$ The series converges.

33) $\sum_{n=1}^{\infty}\frac{2^n}{3^n+1}<\left(\frac{2}{3}\right)^n$ which is geometric and converges because r < 1.

By the Comparison test, our series converges.

35) $\sum_{n=1}^{\infty}n^2e^{-n}$ Use the Ratio test. $\lim_{n\to\infty}\left(\frac{(n+1)^2}{e^{n+1}}\cdot\frac{e^n}{n^2}\right)=\lim_{n\to\infty}\left(\frac{(n+1)^2}{n^2}\cdot\frac{e^n}{e^{n+1}}\right)$

$\lim_{n\to\infty}\left(\frac{(n+1)^2}{n^2}\right)=1$ $\lim_{n\to\infty}\left(\frac{e^n}{e^{n+1}}\right)=\frac{1}{e}$

Since $\frac{1}{e}<1$, the series converges.

37) $\sum_{n=1}^{\infty}\frac{(n+3)!}{3!n!3^n}$ Use the ratio test. $\lim_{n\to\infty}\frac{(n+4)!}{3!(n+1)!3^{n+1}}\cdot\frac{3!n!3^n}{(n+3)!}=$

$\lim_{n\to\infty}\frac{(n+4)!}{(n+3)!}=n+4$ $\lim_{n\to\infty}\frac{3!}{3!}=1$ $\lim_{n\to\infty}\frac{n!}{(n+1)!}=\frac{1}{n+1}$ $\lim_{n\to\infty}\frac{3^n}{3^{n+1}}=\frac{1}{3}$

$\lim_{n\to\infty}\frac{n+4}{3(n+3)}=\frac{1}{3}<1$ The series converges.

39) $\sum_{n=1}^{\infty}\frac{(-2)^n}{3^n}=\sum_{n=1}^{\infty}\left(\frac{-2}{3}\right)^n$ which is geometric with $r=-\frac{2}{3}$ so it converges.

41) $\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$ Use the Ratio test. $\lim_{n\to\infty} \left| \frac{\frac{3^n}{n^3 2^n}}{\frac{3^n}{n^3 2^n}} \right| = \lim_{n\to\infty} \left| \frac{3^{n+1}}{(n+1)^3 2^{n+1}} \cdot \frac{n^3 2^n}{3^n} \right|$

$\lim_{n\to\infty} \frac{3^{n+1}}{3^n} = 3$ $\lim_{n\to\infty} \frac{n^3}{(n+1)^3} = 1$ $\lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}$

$\frac{3}{2} > 1$ so it diverges.

43) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$ Use the Ratio test. $\lim_{n\to\infty} \frac{\frac{(n+1)!}{(2(n+1)+1)!}}{\frac{n!}{(2n+1)!}} = \lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} \right| =$

$\lim_{n\to\infty} \frac{(n+1)!}{n!} = n+1$ $\lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!} = \frac{1}{(2n+3)(2n+2)}$

$\lim_{n\to\infty} \frac{n+1}{(2n+3)(2n+2)} = 0$ which is less than 1

The series converges.

49) $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$ Use partial fractions to create a sum.

$\frac{6}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$

$6 = A(2n+1) + B(2n-1)$

$n = -\frac{1}{2}$ $6 = -2B$ $B = -3$

$n = \frac{1}{2}$ $6 = 2A$ $A = 3$

$\frac{3}{2n-1} - \frac{3}{2n+1}$

$n = 1$ $3 - 1$

$n = 2$ $1 - \frac{3}{5}$

$n = 3$ $\frac{3}{5} - \frac{3}{7}$

The sum approaches 3.

51) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$ This one is hard!!!!

Use partial fractions to create a sum.

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{(n+1)} + \frac{D}{(n+1)^2}$$

$$2n + 1 = An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2$$

Let $n = -1$ $\quad -1 = D$

Let $n = 0$ $\quad 1 = B$

Let $n = 1$ $\quad 3 = 4A + 4B + 2C + D$

$$3 = 4A + 4 + 2C - 1$$

$$0 = 4A + 2C \qquad -2A = C$$

Let $n = 2$ $\quad 5 = 18A + 9B + 12C + 4D$

$$5 = 18A + 9 + 12C - 4$$

$$0 = 18A + 12C \qquad -18A = 12C$$

$$-\frac{3A}{2} = C$$

$$-2A = -\frac{3A}{2} \qquad -4A = -3A \quad A = 0 \qquad C = 0$$

$$\frac{0}{n} + \frac{1}{n^2} + \frac{0}{(n+1)} - \frac{1}{(n+1)^2}$$

$$\frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$n = 1$ $\quad \frac{1}{1} \qquad -\frac{1}{4}$

$n = 2$ $\quad \frac{1}{4} \qquad -\frac{1}{9}$

$n = 3$ $\quad \frac{1}{9} \qquad -\frac{1}{16}$

The series converges to 1.

53) $\sum_{n=1}^{\infty}\left(\frac{1}{\ln{(n+2)}}-\frac{1}{\ln{(n+1)}}\right)$

$n = 1$ $\frac{1}{ln3}-\frac{1}{ln2}$

$n = 2$ $\frac{1}{ln4}-\frac{1}{ln3}$

$n = 3$ $\frac{1}{ln5}-\frac{1}{ln4}$

The series converges to $-\frac{1}{ln2}$.